Dynamic Screening with Liquidity Constraints

Daniel Krähmer and Roland Strausz*

October 8, 2024

Abstract

We consider a dynamic screening model with serially independent types where the agent is short-term liquidity constrained. We model a liquidity constraint as a hard constraint that forces the agent to renege whenever he would suffer a loss from fulfilling the contract terms in a given period. In particular, the violation of a liquidity constraint is a verifiable event that future contract terms can condition on. This verifiability leads to less stringent truth-telling constraints than those considered in the existing literature. We show that the weaker constraints do not affect optimal contracting, however. Moreover, we develop a novel method to study private values settings with continuous types and show that a regularity condition that has analogues in the literature on multi-dimensional screening ensures that the optimal contract is deterministic.

Keywords: Dynamic Screening, Liquidity Constraints, Verifiability, Mean Preserving Spread

JEL: D82, H57

---

*Krähmer: Universität Bonn (kraehmer@hcm.uni-bonn.de); Strausz: Humboldt Universität Berlin (strauszr@hu-berlin.de); We thank Nima Hagpanah, Mathijs Janssen, and Rohit Lamba for useful comments as well as seminar participants in Berlin, Bonn, Duisburg, Jerusalem/Tel-Aviv, Montreal, Peking, and Singapore. In addition we thank the editor and three anonymous referees.

Statement and Declarations: Daniel Krähmer acknowledges financial support from the German Research Foundation (DFG) through Germany's Excellence Strategy EXC 2126/1–390838866 and CRC TR 224. Roland Strausz acknowledges funding by the European Union (ERC, PRIVDIMA, 101096682) and from the German Research Foundation (DFG) through CRC TR 190. Views and opinions expressed are however those of the authors only and do not necessarily reflect those of the European Union or the European Research Council Executive Agency. Neither the European Union nor the granting authority can be held responsible for them.

1 Introduction

A recent literature studies the role of short-term liquidity constraints in dynamic screening models where a procurer (the principal) procures goods or services over multiple periods from a supplier (the agent) whose costs evolve dynamically over time and are the supplier's private information (e.g. Krishna et al., 2013, Mirrokni et al., 2020, Krasikov and Lamba, 2021, Ashlagi et al., 2023). A liquidity constraint is a hard (physical) constraint that captures the fact that, in practice, suppliers are often forced to renege on the contract because they are unable to raise cash for paying short-term bills.[1] By contrast, the classical literature on dynamic screening/mechanism design (e.g., Baron and Besanko, 1984, Battaglini, 2005, Pavan et al., 2014, Esö and Szentes, 2017) neglects such concerns, effectively assuming that the supplier has sufficiently deep pockets to overcome short-term liquidity needs. In this case, optimal contracts exploit this feature and, in fact, impose short-term losses on the supplier. These contracts are thus no longer feasible in situations when the supplier is unable to cope with short-term liquidity needs.

We make two contributions to the literature. First, existing literature focusses on direct revelation mechanisms and imposes liquidity constraints by requiring the mechanism to ensure that the agent gets non-negative utility on the path, that is, when the agent reveals his private information truthfully.[2] This approach is, however, difficult to interpret because it does not ensure that the agent obtains a non-negative periodic utility off the path, that is, when the agent misreports. In fact, the mechanisms that this literature identifies as optimal typically exhibit a *binding* liquidity constraint for some cost type θ so that the liquidity constraint is violated for any cost type $\theta' > \theta$ that misreports to be type θ'.[3]

To clarify these issues, we take serious the idea of the literature that a liquidity constraint is a hard constraint that kicks in whenever the agent would suffer a financial loss when fulfilling the contract terms in the current period. In other words, when illiquid, the agent has no choice

[1] According to one study, about 80% percent of failing small business in the US attribute their bankruptcy to cash flow problems. `https://www.visualcapitalist.com/why-do-businesses-fail/`

[2] Krishna et al. (2013) call these liquidity constraints "non-negative cash flows" and claim in footnote 10 that incentive compatibility and on-path liquidity constraints imply that liquidity constraints are never violated off-path and can therefore be neglected. Krasikov and Lamba (2021) refer to the agent's liquidity constraints as "cash-strapped" and while they state in footnote 22 that "Even if the agent may have misreported in the past, the principal delivers a non-negative stage utility to him if he is *truthful today*" (emphasis added), they however leave unspecified the payoff of an agent who is *not truthful today*. Ashlagi et al. (2023) impose ex post individual rationality, requiring that the agent gets non-negative life-time utility along the (truth-telling) equilibrium path. They note that, in their context, ex post individual rationality is equivalent to requiring that the agent obtains a non-negative per-period utility (along the equilibrium path). However, no restrictions are imposed on life-time or periodic utility off the path.

[3] Hence, this observation is in conflict with footnote 10 in Krishna et al. (2013) claiming that incentive compatibility and on-path liquidity constraints imply that liquidity constraints are satisfied off-path.

but to renege on the contract. Taking this idea to its logical conclusion implies that a violation of liquidity constraints is inherently verifiable.

Moreover, since the agent is forced to renege only because his pockets are empty, we assume that he obtains a payoff equal to (his outside option of) zero.[4] This assumption also captures that the agent as a firm is protected by limited liability. However, as the agent's liquidity is verifiable, the future terms of the contract can be conditioned on whether the current liquidity constraint is violated or not.

Micro-founding liquidity constraints in this way allows us to deduce the contractual feasibility constraints from the underlying physical environment. In particular, the resulting dynamic incentive compatibility constraints account for the possibility to break the liquidity constraint after a deviation from truth-telling. Moreover, since violations of liquidity constraints are verifiable, certain deviations are detectable, and this enlarges the contractual design choices to dissuade them. Specifically, we show that an optimal contract has to satisfy only uni-directional incentive constraints that only prevent an agent from overstating her costs. The reason is that understating one's cost results in a verifiable violation of the liquidity constraint, thus revealing a lie. Even though our uni-directional incentive constraints are weaker than the feasibility constraints posited by the existing literature, a key insight of our paper is that optimal contracts do not, however, exploit the additional slack thus gained. In this sense, our approach validates the literature's approach to impose cash constraints only on but not off the equilibrium path.[5]

Our observation that a liquidity constraint affects the agent's incentive constraint has a counterpart in the literature on mechanism design with quitting or withdrawal rights. In particular, that literature points out that incentive constraints are affected by such rights in that they have to account for the "double deviation" that an agent misreports and subsequently quits (e.g. Matthews and Postlewaite, 1989; Forges, 1999; Compte and Jehiel, 2007, 2009, Krähmer and Strausz, 2015, Bergemann et al., 2020). We stress, however, that despite this similarity, liquidity constraints conceptually differ from quitting or withdrawal rights. This is so because with a quitting or withdrawal right, the agent can strategically decide whether to sustain a loss ex post or

[4]Thus, we abstract from potential non-monetary costs of reneging on the contractual terms, for example reputational costs. As we point out in more detail in Footnote 8, the presence of such costs would possibly allow the principal to induce violations of the liquidity constraint as a screening instrument.

[5]Because our micro foundation implies that an optimal contract has to satisfy only uni-directional incentive constraints, our study of dynamic setting with bankruptcy constraints links to the literature that considers static settings in which such uni-directional incentive constraints exist for exogenous reasons (e.g., Moore, 1984, and Celik, 2006, Krähmer and Strausz, 2024). In line with our finding, this literature shows that, in static settings, these weaker incentive constraints do not give rise to different predictions in settings with private values or, more generally, when the aggregate surplus is monotone in the allocation.

not. By contrast, and as mentioned above, the agent cannot do so in case of a liquidity constraint. Hence, the consideration of quitting or withdrawal rights introduces a moral hazard problem, which does not arise in the case of liquidity constraints. We refer to Krishna et al. (2013, p106) for a more extensive discussion of this distinction.

Our second contribution is to extend the existing literature's analysis of liquidity constraints with two agent types to settings with continuous types. This extension is not straightforward, because with liquidity constraints, the principal's ex ante payoff is a non-linear and non-monotone function of the agent's (future) information rents. Hence, contrary to dynamic screening without liquidity constraints, the problem cannot be reduced to maximizing a virtual surplus representation where allocations are additively separable by type. Consequently, the problem becomes difficult to solve with standard techniques when there are more than two types. For this reason, we develop a novel solution method.

The basic idea behind this method is based on the observation that every dynamic contract induces a continuation value for the agent which, from the principal's perspective, is a random variable, as it depends on the agent's privately known type. A standard argument from dynamic programming implies that the principal's continuation profit is concave in the agent's continuation value. This observation allows us to rank contracts in terms of second order stochastic dominance of the induced continuation value. As a result, we can identify an optimal contract as a contract that, among the set of feasible contracts, displays minimal dispersion in the second order sense. We show that, under a regularity condition, an optimal contract has a simple, deterministic cutoff structure where cost types below a cutoff produce the good and types above the cutoff do not. The regularity condition differs from the more familiar monotone virtual surplus kind of conditions, and also appears in the (static) multi-dimensional screening literature (e.g. Manelli and Vincent (2006)). The connection is that, as in this literature, we write the principal's optimization problem in terms of the agent's (continuation) value rather than the allocation rule.

2 The model

A principal (the buyer, she) and an agent (the seller, he) interact over two periods $\tau = 1, 2$.[6] In each period, the principal seeks to procure one good from the agent. In period τ, the terms of trade are the probability of trade x_τ and a transfer t_τ from the principal to the agent.[7] The

[6] At the end of Section 4, we show that our analysis and results extend to a setting with infinitely many periods.

[7] As is standard, we interpret t_τ as the expected payment $t_\tau^{(0)}(1-x_\tau)+t_\tau^{(1)}x_\tau$, where $t^{(0)}$ (resp. $t^{(1)}$) is the payment when trade does not (resp. does) occur. Alternatively, for a divisible good, we may interpret x_τ as the share of the

principal's valuation for the good is v_τ, and the agent's cost to produce the good is θ_τ. While v_τ is commonly known, θ_τ, the agent's cost type in period τ, is privately known to the agent in period τ, and it is commonly known that θ_τ is distributed with cdf F_τ with support $\Theta_\tau \equiv [\underline{\theta}_\tau, \bar{\theta}_\tau]$ and differentiable pdf f_τ. We assume that θ_1 and θ_2 are stochastically independent. Moreover, we assume that production is only efficient when costs are low enough, i.e. $v_\tau \in (\underline{\theta}_\tau, \bar{\theta}_\tau]$.

The parties have time-separable quasi-linear utilities. That is, under the terms of trade x_τ, t_τ the principal's utility in period τ is $v_\tau x_\tau - t_\tau$, and the agent's utility is $t_\tau - \theta_\tau x_\tau$. A party's overall utility is the sum over the per-period utilities.

The key feature of our paper is that the agent is short-term liquidity constrained. This means that the agent cannot honor the contract in a given period if this would require him to make a loss in this period. We say that the agent is "illiquid" in this case and, as explained in the introduction, the fact that the liquidity constraint is hard implies that being illiquid is a verifiable event. In particular, future contract terms can condition on past liquidity states. Moreover, we assume that when the agent is illiquid, both the agent and the principal receive their reservation utility of zero.[8]

The timing is as follows:

1. At the outset, the principal commits to a long-term contract which specifies the terms of trade over the two periods. If the agent rejects the contract, both parties receive their reservation utility of 0 and the game ends.

2. If the agent accepts, then in period 1, he privately learns θ_1. If $t_1 - \theta_1 x_1 \geq 0$, the agent is liquid and the terms of trade x_1, t_1 are implemented. If $t_1 - \theta_1 x_1 < 0$, the agent is illiquid and both parties receive 0.[9]

3. In period 2, the agent privately learns θ_2. If $t_2 - \theta_2 x_2 \geq 0$, the agent is liquid and the terms of trade x_2, t_2 are implemented. If $t_2 - \theta_2 x_2 < 0$, the agent is illiquid and both parties receive 0.

good traded.

[8]Thus, we assume that the agent is protected by limited liability and cannot be penalized to a level below her outside option when becoming illiquid. This simply reflects that being illiquid means that the agent has empty pockets. In practice, reneging on a contract often comes with additional costs such as reputational costs or the opportunity costs entailed by legal proceedings. To keep the analysis simple, we abstract from the presence of such costs because they would introduce the possibility to "money burning" by way of inducing illiquidity as a screening instrument.

[9]Related to footnote 7, if $x_1 \in (0, 1)$, the agent is illiquid if $t^{(0)} < 0$ and the mechanism does not prescribe trade, as well as if $t^{(1)} - \theta_1 < 0$ and the mechanism does prescribe trade.

Benchmarks: It is useful to contrast our setting to various benchmarks. In the absence of a liquidity constraint, the model corresponds to a traditional dynamic screening model where the agent has an outside option of zero at the contracting stage. If contracting takes place under symmetric information, then the principal can implement the first-best by "selling the firm" to the agent at a price equal to the expected first-best surplus (see, e.g., Harris and Raviv, 1979). The agent will therefore make a loss for high cost realizations. This outcome is therefore not feasible with liquidity constraints.

If contracting takes place after the agent observes θ_1, then an optimal mechanism features a distorted allocation in the first period but implements the first-best in the second period because cost types are independent (see, e.g., Baron and Besanko, 1984). Again, for high cost realizations, the liquidity constraint will be violated in some period. By backloading payments to the agent from the first to the second period, however, the agent's second period payoff can be made non-negative for all θ_2 so that second period liquidity constraints are automatically satisfied. Since these constraints effectively correspond to second period participation constraints, the same outcome can be implemented even if the agent can walk away from the contract after observing θ_2.

Note that in our setting, it is immaterial whether contracting takes place after or before the agent observes θ_1 because the liquidity constraint ensures that the participation constraint is satisfied for all types θ_1 even if contracting takes place under symmetric information (see Sappington, 1983). Finally, if the principal can offer only one-period spot contracts, then he will offer in each period the static one-period second best mechanism, which is a posted price mechanism (see Riley and Zeckhauser, 1983).

Example: To illustrate our analysis, we use the uniform example, where θ_1 and θ_2 are both uniformly distributed over the interval $[0, 1]$, and $v_1 = v_2 = \bar{\theta} = 1$. For this example, trade is efficient for all types and the per-period first-best surplus equals $S^{FB} = S_1^{FB} = S_2^{FB} = \int_0^1 1 - \theta \; d\theta = 1/2$, yielding an aggregate surplus of $S_1^{FB} + S_2^{FB} = 1$. In the static second best, the optimal mechanism is a posted price of $1/2$, yielding the principal a per-period profit of $\Pi^{SB} \equiv (1 - 1/2) * 1/2 = 1/4$ and the agent a per-period second best utility of $U^{SB} \equiv \int_0^{1/2} 1/2 - \theta \; d\theta = 1/8$. Implementing a posted price of $1/2$ for each of the two periods, yields an overall profit of $2\Pi^{SB} = 1/2$ to the principal and an overall utility of $2U^{SB} = 1/4$ to the agent, resulting in aggregate surplus of $3/4$.

In the benchmark case in which there is an interim participation constraint in period 1 but no liquidity constraint, the optimal mechanism implements a posted price of $p = 1/2$ for the first period, and extracts the whole first-best surplus in the second period. This yields an overall profit of $\Pi^{SB} + S^{FB} = 3/4$ to the principal, an overall utility of $U^{SB} = 1/8$ to the agent, resulting in aggregate surplus of $7/8$. □

3 The principal's problem

The principal's objective is to design a contract to maximize her profits. Because the principal has full commitment, the revelation principle applies, implying that an optimal contract is in the class of direct mechanisms where, on the equilibrium path, agent reports his type truthfully in every period (Myerson, 1986). Moreover, because the agent's liquidity in period 1 is verifiable, a mechanism can condition the terms of trade in period 2 on whether the agent was illiquid in period 1 or not. Without loss, we can therefore restrict attention to contracts of the form $(x_1, t_1, x_2^L, t_2^L, x_2^I, t_2^I)$, where

$$(x_1, t_1) : [\underline{\theta}_1, \bar{\theta}_1] \to [0,1] \times \mathbb{R}, \qquad (x_2^\ell, t_2^\ell) : [\underline{\theta}_1, \bar{\theta}_1] \times [\underline{\theta}_2, \bar{\theta}_2] \to [0,1] \times \mathbb{R}, \tag{1}$$

where $\ell \in \{I, L\}$ indicates whether the agent was illiquid ($\ell = I$) or liquid ($\ell = L$) in period 1.

To state the incentive compatibility constraints, we denote for $\ell \in \{I, L\}$ the agent's expected period 2 utility from a report $\hat{\theta}_1$, conditional on truthfully reporting in period 2, by

$$U^\ell(\hat{\theta}_1) = \int_{\underline{\theta}_2}^{\bar{\theta}_2} \max\{0, t_2^\ell(\hat{\theta}_1, \theta_2) - \theta_2 x_2^\ell(\hat{\theta}_1, \theta_2)\}\, dF_2(\theta_2). \tag{2}$$

Moreover, let

$$\Theta_1^L = \{\theta_1 \mid t_1(\theta_1) - \theta_1 x_1(\theta_1) \geq 0\} \tag{3}$$

be the set of period 1 types who are liquid in period 1 under a given mechanism.

Definition 1 *A contract $(x_1, t_1, x_2^L, t_2^L, x_2^I, t_2^I)$ is feasible if:*
(i) It is incentive compatible in period 2, that is, for $\ell \in \{I, L\}$:[10]

$$\max\{0, t_2^\ell(\theta_1, \theta_2) - \theta_2 x_2^\ell(\theta_1, \theta_2)\} \geq \max\{0, t_2^\ell(\theta_1, \hat{\theta}_2) - \theta_2 x_2^\ell(\theta_1, \hat{\theta}_2)\} \quad \forall \theta_1, \theta_2, \hat{\theta}_2. \tag{4}$$

[10] The revelation principle for dynamic games requires truthful reporting in period 2 only after a truthful report in period 1 (see Myerson, 1986). In our context, where types are independent, the support of period 2 types is "non-shifting", that is, is independent of the period 1 type. It then follows with standard arguments that if truth-telling in period 2 is optimal for the agent after telling the truth in period 1, then it is so after any report in period 1.

(ii) It is incentive compatible in period 1, that is:

⋄ *For all $\theta_1 \in \Theta_1^L$, we have:*

$$t_1(\theta_1) - \theta_1 x_1(\theta_1) + U^L(\theta_1) \geq t_1(\hat{\theta}_1) - \theta_1 x_1(\hat{\theta}_1) + U^L(\hat{\theta}_1) \quad \forall \hat{\theta}_1 : t_1(\hat{\theta}_1) - \theta_1 x_1(\hat{\theta}_1) \geq 0, \quad (5)$$

$$t_1(\theta_1) - \theta_1 x_1(\theta_1) + U^L(\theta_1) \geq U^I(\hat{\theta}_1) \quad \forall \hat{\theta}_1 : t_1(\hat{\theta}_1) - \theta_1 x_1(\hat{\theta}_1) < 0, \quad (6)$$

⋄ *For all $\theta_1 \notin \Theta_1^L$, we have:*

$$U^I(\theta_1) \geq t_1(\hat{\theta}_1) - \theta_1 x_1(\hat{\theta}_1) + U^L(\hat{\theta}_1) \quad \forall \hat{\theta}_1 : t_1(\hat{\theta}_1) - \theta_1 x_1(\hat{\theta}_1) \geq 0, \quad (7)$$

$$U^I(\theta_1) \geq U^I(\hat{\theta}_1) \quad \forall \hat{\theta}_1 : t_1(\hat{\theta}_1) - \theta_1 x_1(\hat{\theta}_1) < 0. \quad (8)$$

Part (i) of the definition captures the truth-telling constraints for the agent in period 2, explicitly taking into account the agent's liquidity in period 2 in period 2 occurs whenever the terms of trade would impose a loss on the agent. Similarly, part (ii) describes the truth-telling constraints for the agent in period 1. This requires a distinction between four cases, depending on how both truth-telling and lying affects the agent's liquidity in period 1.

The principal's problem is thus to select a feasible contract that maximizes her profits

$$\int_{\Theta_1^L} \left[v_1 x_1(\theta_1) - t_1(\theta_1) + \int_{\Theta_2^{L,L}(\theta_1)} v_2 x_2^L(\theta_1, \theta_2) - t_2^L(\theta_1, \theta_2)\, dF_2(\theta_2) \right] dF_1(\theta_1) \quad (9)$$

$$+ \int_{\Theta_1 \setminus \Theta_1^L} \left[0 + \int_{\Theta_2^{I,L}(\theta_1)} v_2 x_2^I(\theta_1, \theta_2) - t_2^I(\theta_1, \theta_2)\, dF_2(\theta_2) \right] dF_1(\theta_1), \quad (10)$$

where

$$\Theta_2^{\ell,L}(\theta_1) \equiv \{\theta_2 \in \Theta_2 \mid t_2^\ell(\theta_1, \theta_2) - \theta_2 x_2^\ell(\theta_1, \theta_2) \geq 0\} \quad (11)$$

denotes the set of types θ_2 who are liquid in period 2 given their liquidity state $\ell \in \{L, I\}$ in period 1.

To solve the principal's problem, we first show that it is without loss to focus on contracts with the property that the agent is liquid on the equilibrium path where the agent tells the truth.

The intuition is simply that the outcome when the agent is illiquid is equivalent to not trading the good ($x = 0$) and making no payments ($t = 0$), which keeps the agent just liquid. Thus, the outcome of a mechanism γ where the agent becomes illiquid on the equilibrium path can be replicated by the mechanism which differs from γ only in that it specifies no trade and zero payments for any contingency where the agent becomes illiquid on path under γ.

Second, it is without loss to focus on mechanisms in which the agent exactly breaks even in the first period, and backloads any potential profit for the agent in that it accrues only in the second period.[11] The reason is that if the agent were to make a profit in the first period, the principal could deduct it from the agent's first period 1 payments and pay it out in period 2 instead. This would not affect the principal's profit and would maintain truth-telling incentives for which only total payments matter.

We summarize these considerations in the next lemma.

Lemma 1 *For any feasible contract there is a payoff-equivalent feasible contract* $(x_1, t_1, x_2^L, t_2^L, x_2^I, t_2^I)$ *with the following properties:*

- *The agent exactly breaks even, and is never illiquid in period 1 (on path):*

$$t_1(\theta_1) - \theta_1 x_1(\theta_1) = 0 \quad \textit{for all } \theta_1. \tag{12}$$

- *The agent is never illiquid in period 2 (on path):*

$$t_2^L(\theta_1, \theta_2) - \theta_2 x_2^L(\theta_1, \theta_2) \geq 0 \quad \textit{for all } \theta_1, \theta_2. \tag{13}$$

- *After the off-path event that the agent becomes illiquid in period 1, the relationship is terminated:*

$$x_2^I(\theta_1, \theta_2) = t_2^I(\theta_1, \theta_2) = 0 \quad \textit{for all } \theta_1, \theta_2. \tag{14}$$

Lemma 1 implies that we can find an optimal contract in the class of feasible contracts that satisfy (12)-(14). Since properties (12) and (14) pin down t_1, x_2^I, and t_2^I, we are actually left to determine only the triple (x_1, x_2^L, t_2^L). We therefore introduce the following definition.

Definition 2 *A triple* (x_1, x_2, t_2) *with* $x_1 : \Theta_1 \to [0,1]$, $x_2 : \Theta_1 \times \Theta_2 \to [0,1]$, *and* $t_1 : \Theta_1 \times \Theta_2 \to \mathbb{R}$ *is called a backloaded contract if*

$$L_2: \qquad t_2(\theta_1, \theta_2) - \theta_2 x_2(\theta_1, \theta_2) \geq 0 \quad \forall \theta_1, \theta_2. \tag{15}$$

A backloaded contract uniquely induces a contract $(x_1, t_1, x_2^L, t_2^L, x_2^I, t_2^I)$ with the properties (12)-(14) by setting $t_1 = \theta_1 x_1$, $x_2^L = x_2$, $t_2^L = t_2$, and $x_2^I = t_2^I = 0$. For a backloaded contract, we

[11] This argument also appears in Ashlagi et al. (2023).

write

$$U(\theta_1) = \int_{\underline{\theta}_2}^{\bar{\theta}_2} t_2(\theta_1, \theta_2) - \theta_2 x_2(\theta_1, \theta_2)\, dF_2(\theta_2) \tag{16}$$

for the agent's expected period 2 utility. The next lemma characterizes when a backloaded contract is feasible.

Lemma 2 *A backloaded contract (x_1, x_2, t_2) induces a feasible contract $(x_1, t_1, x_2^L, t_2^L, x_2^I, t_2^I)$ if and only if*

$$IC_2: \qquad t_2(\theta_1, \theta_2) - \theta_2 x_2(\theta_1, \theta_2) \geq t_2(\theta_1, \hat{\theta}_2) - \theta_2 x_2(\theta_1, \hat{\theta}_2) \quad \forall \theta_1, \theta_2, \hat{\theta}_2 \tag{17}$$

$$IC_1: \qquad U(\theta_1) \geq (\hat{\theta}_1 - \theta_1) x_1(\hat{\theta}_1) + U(\hat{\theta}_1) \quad \forall \theta_1 < \hat{\theta}_1 \tag{18}$$

$$IC_1^0: \qquad U(\theta_1) \geq U(\hat{\theta}_1) \quad \forall \hat{\theta}_1 \in \Theta_1^0, \forall \theta_1 \in \Theta_1, \tag{19}$$

where $\Theta_1^0 = \{\theta \in \Theta_1 | x_1(\theta) = 0\}$ is the set of types who do not trade in period 1.

Constraint IC_2 corresponds to the period 2 truth-telling constraints (4). The more interesting constraint is IC_1 which corresponds to the period 1 truth-telling constraints.[12]

The novelty is that IC_1 only requires that the agent does not report a higher type than his true type, but not that he does not report a lower type. The reason for this asymmetry is that if the agent reported lower costs in the first period, then, because any cost type breaks even in period 1, the agent would become illiquid after such a lie. But, a violation of the liquidity constraint does not occur on path and is verifiable. Consequently, such a lie would be detected, and under a backloaded mechanism, the relationship would be terminated. This prospect is enough to dissuade the agent from understating his costs, and extra incentives are not needed to induce truth-telling.

A subtlety is however that the agent can become illiquid only when some production takes place. As a result, the previous reasoning applies only to types $\hat{\theta}_1$, who actually produce in period 1. The constraint IC_1^0 takes care of this by explicitly requiring that for types $\hat{\theta}_1$ with $x_1(\hat{\theta}_1) = 0$, the truth-telling constraints have to hold in both directions, as the verifiability of the agent's liquidity has no bite in this case.

We can now re-state the principal's problem as selecting an optimal backloaded contract. Under a backloaded contract, the principal's period 1 profit equals $v_1 x_1(\theta_1) - \theta_1 x_1(\theta_1)$ for all

[12]To see that IC_1 replaces the period 1 truth-telling constraints (5)-(8) note that because under a backloaded contract the agent is never illiquid in period 1 for any θ_1, constraints (6), (7) and (8) are all redundant, and the only relevant constraint is (5), which now has to hold for all θ_1, leading to IC_1.

θ_1, because the agent breaks even in period 1. Moreover, her profit in period 2 is equal to $v_2 x_2(\theta_1, \theta_2) - t_2(\theta_1, \theta_2)$ for all (θ_1, θ_2), because the agent is never illiquid in period 2. Thus, the principal's problem is

$$P: \qquad \sup_{x_1, x_2, t_2} \int_{\underline{\theta}_1}^{\bar{\theta}_1} \int_{\underline{\theta}_2}^{\bar{\theta}_2} v_1 x_1(\theta_1) - \theta_1 x_1(\theta_1) + v_2 x_2(\theta_1, \theta_2) - t_2(\theta_1, \theta_2)\, dF_2(\theta_2)\, dF_1(\theta_1)$$
$$s.t. \quad IC_2, L_2, IC_1, IC_1^0$$

where IC_2, IC_1, and IC_1^0 are the feasibility constraints, and L_2 ensures that the contract is a backloaded contract. Note that the constraint IC_1^0 leads to the technical complication that the feasibility set is not closed.[13]. For this reason, the principal's objective does not necessarily take on a maximum. We will address this issue explicitly.

Intuitively, for given θ_1, the principal faces the standard (intra-temporal) rent-efficiency trade-off in period 2, because she has to grant low cost types an information rent for truth-telling due to the presence of the liquidity constraint L_2. Moreover, because a backloaded contract uses the *expected* period 2 information rent for incentivizing low cost types in period 1 to reveal the truth, the principal also faces an inter-temporal trade-off between maximizing profits in period 1 and 2.

Before solving problem P, we note that the liquidity constraint L_2 is formally equivalent to a period 2 participation constraint. Hence, problem P corresponds to a two-period dynamic mechanism design problem with second period participation constraints, but with the novelty that the first period incentive constraints IC_1 are asymmetric in that they only require lower types not to mimic higher types, and the constraints IC_1^0 only apply to reports that imply no production. To the extent that those modified incentive constraints result from the first period liquidity constraint, the latter is the key constraint.

More formally, the incentive constraints IC_1 and IC_1^0 are precisely what distinguishes our problem from the kind of problem that arises in the literature that imposes liquidity constraints only on the path (see footnote 2). In particular, this literature imposes the period 2 liquidity constraint (4), but different from us, imposes a period 1 liquidity constraint $t_1(\theta_1) - \theta_1 x_1(\theta_1) \geq 0$ for all θ_1, and requires the period 1 incentive compatibility constraint (5) to hold *for all* $\theta_1, \hat{\theta}_1$,

[13] To see this, let $\Theta_1 = [0,1]$ and consider sequence of contracts with (x_1^n, U_1^n) given by

$$x_1^n(\theta_1) = \begin{cases} 1/n & if \quad \theta_1 \in [0, 1/2) \\ 1 & if \quad \theta_1 \in [1/2, 1] \end{cases}, \qquad U_1^n(\theta_1) = 1 - \theta_1. \tag{20}$$

Note that for every n, $\Theta_1^0 = \emptyset$ so that IC_1^0 is redundant, and it is easy to check that IC_1 is satisfied. Thus, every element in the sequence is feasible. However, the limit contract, as $n \to \infty$, is not feasible, because in the limit, $\Theta_1^0 = [0, 1/2)$ and so IC_1^0 is violated for any pair $(\theta_1, \hat{\theta}_1)$ with $\hat{\theta}_1 < \max\{\theta_1, 1/2\}$

and does not condition the agent's continuation value on the agent's liquidity in period 1, and thus does not consider the constraints (6) to (8). Like in our case, it is then without loss to focus on backloaded contracts, and the resulting problem for the principal differs from problem P only in that constraint IC_1^0 is missing and IC_1 is replaced by its bi-directional counterpart that requires IC_1 to hold *for all* $\theta_1, \hat{\theta}_1$.

4 Solution to the principal's problem

We solve the principal's problem by the well-known technique in dynamic programming to reduce the dynamic problem P to a sequence of static problems (Spear and Srivastava, 1987, Thomas and Worrall, 1990). Consequently, we proceed in two steps. In the first step, we solve for optimal period 2 terms of trade (x_2^U, t_2^U) that promise the agent a certain exogenously given expected period 2 utility U. In the second step, we then solve for an optimal period 1 allocation x_1 and an optimal continuation value U taking as given optimal period 2 terms of trade (x_2^U, t_2^U) from step 1 that supply the agent with U.

Step 1: Optimal period 2 terms of trade

Note first that the principal can promise any positive utility $U \geq 0$. Indeed, by L_2, the principal cannot promise a negative utility while she can offer any utility $U \geq 0$ by, for example, offering the terms of trade $(x_2, t_2) = (0, U)$. Thus, the set of feasible promised utilities is $\{U \mid U \geq 0\}$.

For a given report θ_1, the principal's problem to optimally promise $U \geq 0$ is

$$P_2: \quad \Pi(U) \equiv \max_{x_2, t_2} \quad \int_{\underline{\theta}_2}^{\bar{\theta}_2} v_2 x_2(\theta_1, \theta_2) - t_2(\theta_1, \theta_2)\, dF_2(\theta_2) \quad s.t \tag{21}$$

$$IC_2: \quad t_2(\theta_1, \theta_2) - \theta_2 x_2(\theta_1, \theta_2) \geq t_2(\theta_1, \hat{\theta}_2) - \theta_2 x_2(\theta_1, \hat{\theta}_2) \quad \forall \theta_2, \hat{\theta}_2 \tag{22}$$

$$L_2: \quad t_2(\theta_1, \theta_2) - \theta_2 x_2(\theta_1, \theta_2) \geq 0 \quad \forall \theta_2 \tag{23}$$

$$PK: \quad \int_{\underline{\theta}_2}^{\bar{\theta}_2} t_2(\theta_1, \theta_2) - \theta_2 x_2(\theta_1, \theta_2)\, dF_2(\theta_2) = U. \tag{24}$$

While the constraints IC_2 and L_2 carry over from problem P, the constraint PK ensures that the agent receives his promised utility U.

Problem P_2 corresponds to a static monopoly problem where the agent has an interim outside option of 0 after learning θ_2 (as reflected by L_2), and an ex-ante outside option of U before learning θ_2 (as reflected by PK). The solution is well-known from Samuelson (1984); for details see our Remark 1 below. We state the features that will be key for our purposes in the next lemma. In order to avoid uninteresting case distinctions, we impose the following mild condition.[14]

Assumption 1: The second best solution (x_2^{SB}, t_2^{SB}) to the relaxed version of P_2 where PK is missing is unique.

Given Assumption 1, the utilities in the second best solution for both the principal and the agent are unique and we denote them, respectively, by Π_2^{SB} and U_2^{SB}. Moreover, we denote the surplus associated with the period 2 first-best allocation $x_2^{FB}(\theta_1, \theta_2) \equiv \mathbf{1}_{[\underline{\theta}_2, \min\{v_2, \bar{\theta}_2\}]}(\theta_2)$ by[15]

$$S_2^{FB} = \int_{\underline{\theta}_2}^{\min\{v_2, \bar{\theta}_2\}} v_2 - \theta_2 \, dF_2. \tag{26}$$

Clearly, $U_2^{SB} \in (0, S_2^{FB})$.

Lemma 3 *The value of problem P_2 as a function of U, $\Pi(U)$, is concave in U with $\Pi(0) = \Pi(S_2^{FB}) = 0$ and attains a unique maximum Π_2^{SB} at U_2^{SB}, that is, $\Pi(U_2^{SB}) = \Pi_2^{SB}$.*

The concavity of the value follows from a standard mixing argument. Specifically, given two promises U' and U'', the principal can promise the agent the convex mixture $\bar{U} = \alpha U' + (1-\alpha)U''$ by appropriately randomizing between the optimal terms of trade for U' and U''. This would yield the principal a profit $\alpha\Pi(U_2') + (1-\alpha)\Pi(U_2'')$, but by re-optimizing, the principal can promise $\bar{U}$ at a higher profit, implying that Π is concave.

To see why $\Pi(0) = 0$, note that IC_2 and L_2 imply that the only way to provide the agent with expected utility $U = 0$ is through no trade ($x_2 = t_2 = 0$), resulting in zero profits for the principal. To see that $\Pi(S_2^{FB}) = 0$, observe that the principal's profit Π is the total surplus generated minus the utility U supplied to the agent. Hence, if the principal promises the entire first-best surplus to the agent, $U = S_2^{FB}$, she cannot make a strictly positive profit. But she can at least guarantee herself zero profits by selecting terms of trade that generate the first-best surplus. Therefore,

[14] It is well-known (e.g. Riley and Zeckhauser, 1983) that in the absence of PK, the solution to P_2 can be implemented by a posted price p given by

$$p \in \arg\max_{\tilde{p}} \int_{\underline{\theta}_2}^{\tilde{p}} v_2 - \theta_2 - \frac{F_2(\theta_2)}{f_2(\theta_2)} \, dF_2(\theta_2). \tag{25}$$

Notice that the right hand side is generically a singleton in the sense that whenever it is not a singleton, a slight perturbation of $\frac{F_2(\theta_2)}{f_2(\theta_2)}$ would remove all but one solution. Hence, Assumption 1 is mild in that it holds generically.

[15] Given a set A, the indicator function $\mathbf{1}_A(a)$ is 1 if $a \in A$ and 0 otherwise.

$\Pi(S_2^{FB}) = 0$. Finally, that the principal's profit is uniquely maximized at U^{SB} follows directly from the definition of the second best and Assumption 1.

Remark 1 (Period 2 implementation) We briefly discuss how the period 2 contract can be implemented. If the principal promises more than the second best surplus, $U \geq S_2^{FB}$, then it follows from the proof of Lemma 3 that the optimal contract displays the first-best allocation $x_2^{FB}(\theta_1, \theta_2)$. The intuition is that when guaranteeing the agent a utility exceeding the first-best level S_2^{FB}, the principal does not face the standard rent-efficiency trade-off anymore. Now, if the principal promises exactly $U = S_2^{FB}$, this can be indirectly implemented by an offer from the principal to procure the good at a price of v_2. If the principal promises strictly more, $U > S_2^{FB}$, then an optimal contract can be indirectly implemented by a two-part tariff. In particular, the principal makes an unconditional payment $U - S_2^{FB}$ and offers to procure the good at a price of v_2. This two-part tariff yields her an expected profit $\Pi(U) = S_2^{FB} - U$.

If, on the other hand, $U < S_2^{FB}$, it follows from Samuelson (1984) that an optimal trading probability is of the form

$$x_2(\theta_1, \theta_2) = \begin{cases} 1 & if \quad \theta_2 \in [0, \theta_2'] \\ \xi & if \quad \theta_2 \in (\theta_2', \theta_2''] \\ 0 & else \end{cases} \tag{27}$$

for some $\xi \in [0,1]$, $\underline{\theta} \leq \theta_2' \leq \theta_2'' < v_2$ which all depend on U. If $\xi = 0$, the optimal contract can be implemented by an offer from the principal to procure the good at price θ_2'. If $\xi > 0$, the optimal contract can be implemented by a menu of three options for the agent: to not produce the good at a price of 0; to produce a "fraction" ξ of the good for a price of $\xi\theta_2'$,[16] or to produce the good at price θ_2''.

Whether ξ is strictly positive or not, depends on the distribution F_2 and on the size of U. For the special case that the hazard rate F_2/f_2 is increasing, we have that $\xi = 0$ for all U so that the optimal contract can be implemented with a posted price.[17]

Example: For our uniform example, the hazard rate $F_2(\theta_2)/f_2(\theta_2) = \theta_2$ is increasing so that, as noted in Remark 1, a posted price in period 2 is optimal. In particular, the optimal contract for $U \in [0, S_2^{FB}] = [0, 1/2]$ corresponds to a posted price p_2 which maximizes $p_2(v - p_2)$ subject to the promise keeping constraint $\int_0^{p_2} p_2 - \theta \, d\theta = U$. This constraint simplifies to $p_2^2/2 = U$ and

[16] For an indivisible good the contract randomizes between trade and no trade and the agent is payed θ_2' if trade is the outcome.

[17] If F_2/f_2 is increasing the posted price is unique and Assumption 1 holds.

therefore pins down $p_2(U) = \sqrt{2U}$. The resulting profit is $\Pi(U) = p_2(v - p_2) = \sqrt{2U}(1 - \sqrt{2U}) = \sqrt{2U} - 2U$. Moreover, for $U > 1/2$, the optimal contract is efficient and corresponds to a posted price p_2 that satisfies $p_2 - \int_0^1 \theta \, d\theta = U$, that is, $p_2 = U + 1/2$. Moreover, we have $\Pi(U) = 1/2 - U$. Taken together, we thus have:

$$\Pi(U) = \begin{cases} \sqrt{2U} - 2U & if \quad U \leq 1/2; \\ 1/2 - U & if \quad U > 1/2. \end{cases} \tag{28}$$

Note that $\Pi(U)$ is not only continuous but also differentiable at $U = 1/2$.

Step 2: Optimal period 1 terms of trade

Step 1 allow us to re-write the principal's problem P as a static maximization problem over the period 1 terms of trade x_1 and the agent's promised utility U. More specifically, any combination $(x_1(\theta_1), U(\theta_1))$ with $U(\theta_1) \geq 0$ corresponds to a backloaded contract $(x_1(\theta_1), x_2(\theta_1, \cdot), t_2(\theta_1, \cdot))$ where $(x_2(\theta_1, \cdot), t_2(\theta_1, \cdot))$ is the solution to P_2 with $U = U(\theta_1)$. We refer to (x_1, U) as a *reduced* backloaded contract (and simply as backloaded contract if there is no risk of confusion). Clearly, only contracts corresponding to reduced backloaded contracts can be optimal.

Recall that under a backloaded contract, the agent breaks even in period 1, that is, $t_1(\theta_1) = \theta_1 x_1(\theta_1)$. Consequently, the principal receives the profit $v_1 x_1(\theta_1) - \theta_1 x_1(\theta_1)$ in period 1 and the profit $\Pi(U(\theta_1))$ in period 2. Suppressing the time index for period 1, we can therefore rewrite the principal's problem as

$$P_1: \quad \sup_{x,U} \quad \int_{\underline{\theta}}^{\bar{\theta}} [v - \theta] x(\theta) + \Pi(U(\theta)) \, dF(\theta) \quad s.t \tag{29}$$

$$IR: \quad U(\theta) \geq 0 \quad \forall \theta \tag{30}$$

$$IC: \quad U(\theta) \geq (\hat{\theta} - \theta) x(\hat{\theta}) + U(\hat{\theta}) \quad \forall \theta < \hat{\theta} \tag{31}$$

$$IC^0: \quad U(\theta) \geq U(\hat{\theta}) \quad \forall \hat{\theta} \in \Theta^0, \forall \theta \in \Theta \tag{32}$$

$$UG: \quad x(\theta) \in [0, 1] \quad \forall \theta. \tag{33}$$

The constraint IR is simply the feasibility constraint that the agent's expected utility cannot be negative, and the constraints IC and IC^0 are inherited from the original formulation of P. To deal with the problem that the feasibility set is not closed, we solve a relaxed version where we drop

IC^0 leading to the following problem:

$$P_1': \qquad \max_{x,U} \qquad \int_{\underline{\theta}}^{\bar{\theta}} [v-\theta]x(\theta) + \Pi(U(\theta))\, dF(\theta) \quad \text{s.t} \quad IR, IC, UG. \tag{34}$$

Problem P_1' looks similar to a standard monopoly problem where $U(\theta)$ is agent type θ's information rent, and constraint IR corresponds to a standard (interim) participation constraint. There are, however, two important differences.

First, unlike in the static monopoly problem with transferable utility, the principal's objective is not linear in the agent's information rent. This is due to the period 2 liquidity constraint which results in a rent-efficiency trade-off in period 2. In fact, if the principal did not face a liquidity constraint in period 2, her objective would be linear in the information rent because she would then optimally implement the first-best allocation in period 2 and could award the agent any (possibly negative) level of rent through an appropriate transfer.

To shed more light on the principal's costs of providing incentives, recall that Π is single-peaked with a maximum at U^{SB}. Therefore, if the period 1 type were publicly known, the principal would maximize the objective by picking an efficient $x(\theta)$ and setting $U(\theta)$ equal to the second best information rent U^{SB}. But since the period 1 type θ is private information, the principal has to create a spread in the information rents and award a higher information rent to low cost than to high cost types to induce the former to report truthfully. As a result, the cost of providing incentives through promising a certain information rent U in period 2 is not monotone in U. For example, "punishing" the agent with a zero information rent in period 2 is extremely costly, since $\Pi(0) = 0$ means that the principal has to sacrifice the entire surplus in period 2. Likewise, to reward the agent with a rent higher than U^{SB}, then because Π is maximal at U^{SB}, the principal has to give the agent a surplus share that exceeds the second best share of the surplus in period 2.

Second, the incentive compatibility constraints IC are uni-directional, only requiring that the agent does not report a less efficient type. In contrast to the setting with bi-directional incentive constraints, the uni-directional constraints prevent us from employing familiar solution techniques that are based on the characterization of incentive compatibility in terms of monotonicity of the trading probability and revenue equivalence. In fact, it is easy to see that IC does not even imply that $x(\theta)$ is monotone.[18] To address this issue, the next lemma provides necessary

[18] Analyzing a screening problem with uni-directional incentive constraints and discrete types, Celik (2006) makes the same observation. His techniques for solving the subsequent problem do not apply to our framework with continuous types.

conditions for IC that allow us to relax problem P_1'.

Lemma 4 *If (x,U) satisfies IC, then it satisfies the two following conditions:*

M: U is decreasing.

IC_L: $U'(\theta) \leq -x(\theta)$ for all θ where the derivative exists.

Property M is straightforward and simply reflects that lower cost types can guarantee themselves at least the utility of a higher cost type by pretending to be that type. As to condition IC_L, note first that because U is decreasing, U is differentiable almost everywhere. Recall that in the standard case where IC is required for *all* reports $\hat{\theta}$, the derivative of the agent's utility is actually pinned down by the allocation x. In our case, where IC is required only for reports $\hat{\theta} > \theta$, it is only necessary that the derivative of the agent's utility is bounded by the allocation x.

The lemma implies that we obtain a relaxed version of P_1' if we replace IC with the monotonicity condition M and the "localized" condition IC_L:

$$R_1: \qquad \max_{x,U} \quad \int_{\underline{\theta}}^{\bar{\theta}} [v-\theta]x(\theta) + \Pi(U(\theta))\, dF(\theta) \quad s.t \quad IR, M, IC_L, UG \tag{35}$$

We now solve R_1 and then show that its solution also solves P_1'. We proceed in two steps. We first show that at a solution to R_1, trade never happens if it is inefficient, and the constraint IC_L is binding. In the second step, we use these properties to establish a solution to R_1. To establish the first step, let Φ be the (non-empty) feasible set for problem R_1. We then obtain the following result.

Lemma 5 *Let $(\tilde{x},\tilde{U}) \in \Phi$. Then there is $(x,U) \in \Phi$ which delivers the principal a (weakly) higher profit than $(\tilde{x},\tilde{U})$ and has the following properties:*

(i) If $v < \bar{\theta}$, then $x(\theta) = 0$ for all $\theta > v$.

(ii) $U'(\theta) = -x(\theta)$ for all θ.

The first part makes the familiar point that an optimal contract induces a downward distortion. To understand the second part, recall that Π is concave with a maximum at U^{SB}. For a given trading probability x, the principal therefore seeks to choose U as closely as possible to U^{SB} while maintaining the incentive compatibility requirement that $U'(\theta) \leq -x(\theta)$. Thus, an optimal choice of U is maximally flat, implying that $U'(\theta) = -x(\theta)$.

We emphasize that although property (ii) corresponds to the revenue equivalence property from standard screening models where IC is required for *all* reports $\hat{\theta}$, in our setting, property (ii) expresses an optimality rather than a feasibility condition.

In standard screening models, property (ii) is useful, because it pins down the agent's utility U as an integral over the trading probability x. If, in addition, Π is linear, an integration by parts argument can be used to replace U in the objective function of (35), and the problem can then be solved by point-wise maximization over $x(\theta)$. In our case, because Π is concave, this approach does not work.

Our alternative approach is to instead use property (ii) to replace the trading probability x by the agent's utility function U in the objective function of (35) and then maximize over U. This allows us to show that an optimal contract is in the class of *cutoff-contracts* where the good is traded if and only if that agent's cost is below a cutoff $\theta_0 \in [\underline{\theta}.\bar{\theta}]$.

Definition 3 *A cutoff-contract (x, U) is characterized by two parameters: a cutoff $\theta_0 \in [\underline{\theta}, \bar{\theta}]$ and an intercept $U_0 \geq \theta_0 - \underline{\theta}$ such that*

$$x(\theta) = \begin{cases} 1 & if \quad \theta \leq \theta_0 \\ 0 & else \end{cases} , \quad U(\theta) = \begin{cases} U_0 - (\theta - \underline{\theta}) & if \quad \theta \leq \theta_0 \\ U_0 - (\theta_0 - \underline{\theta}) & else. \end{cases} \tag{36}$$

We denote by Λ the set of cutoff contracts. Clearly, $\Lambda \subset \Phi$. We now state the main result of this section that, under a regularity condition, a cutoff-contract is a solution to the relaxed problem R_1.

Proposition 1 *Let $(v - \theta)\frac{f'(\theta)}{f(\theta)}$ be increasing on the range $[\underline{\theta}, \min\{v, \bar{\theta}\}]$. Consider $(\tilde{x}, \tilde{U}) \in \Phi$. Then there is a cutoff-contract $(x, U) \in \Lambda$ which delivers a (weakly) higher profit than $(\tilde{x}, \tilde{U})$.*

While we prove the proposition in the appendix, the underlying logic is best understood in the context of our uniform example. Note that the uniform example satisfies the regularity condition trivially, as $f'(\theta) = 0$.

Example: Consider some $(\tilde{x}, \tilde{U}) \in \Phi$. As indicated earlier, we can use part (ii) of Lemma 5 to replace $\tilde{x}$ by $\tilde{U}'$ in the objective of (35). Using integration by parts and $v = 1$, the objective then rewrites as

$$\int_{\underline{\theta}}^{\bar{\theta}} [v - \theta]\tilde{x}(\theta) + \Pi(\tilde{U}(\theta))\, dF(\theta) = \int_0^1 -[v - \theta]\tilde{U}'(\theta) + \Pi(\tilde{U}(\theta))\, d\theta \tag{37}$$

$$= \tilde{U}(0) + \int_0^1 \tilde{U}(\theta)\, d\theta + \int_0^1 \Pi(\tilde{U}(\theta))\, d\theta. \tag{38}$$

We now construct a function U belonging to a cutoff-contract for which expression (38) is at least as large as for $\tilde{U}$. To do so, note that Lemma 5 implies that $\tilde{U}$ is a decreasing continuous

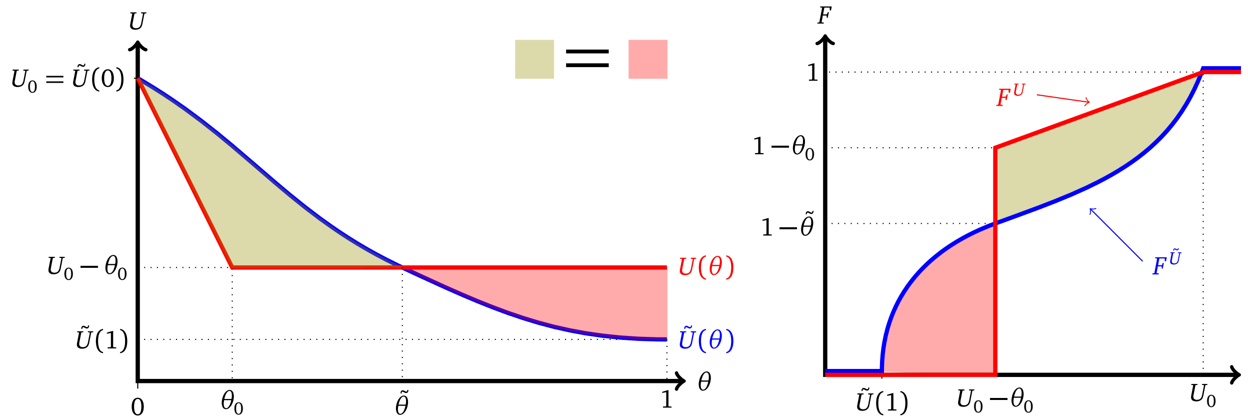

Figure 1: The left panel illustrates, given $\tilde{U}$ and that θ is uniformly distributed over $[0,1]$, the construction of the cutoff contract $U(.)$ such that $U_0 = \tilde{U}(0)$ and $\int_0^1 U(\theta)\,d\theta = \int_0^1 \tilde{U}(\theta)\,d\theta$. The right panel shows the associated probability distributions F^U and $F^{\tilde{U}}$ of U and $\tilde{U}$ in utility space. $F^{\tilde{U}}$ is a mean preserving spread of F^U.

function with a slope between -1 and 0. Therefore, because under a cutoff-contract, U has slope -1 up to the cutoff θ_0 and then slope 0, an intermediate value argument implies that we can find U so that

$$U_0 = \tilde{U}(0), \qquad \int_0^1 U(\theta)\,d\theta = \int_0^1 \tilde{U}(\theta)\,d\theta. \tag{39}$$

In particular, there is a $\tilde{\theta} \in [0,1]$ so that

$$U(\theta) \le \tilde{U}(\theta) \text{ for } \theta \le \tilde{\theta} \text{ and } U(\theta) \ge \tilde{U}(\theta) \text{ for } \theta \ge \tilde{\theta}. \tag{40}$$

The first panel of Figure 1 illustrates the construction graphically.

By (39), the first two terms in (38) are the same for U and $\tilde{U}$. The key idea to analyze the third term in (38) is to interpret the agent's utility as a random variable which induces a probability distribution in utility space (the pushforward). Formally, and as illustrated in the second panel of Figure 1, the distributions induced by $\tilde{U}$ and U correspond to the cumulative distribution functions

$$F^{\tilde{U}}(u) = Pr(\theta \in [0,1] : \tilde{U}(\theta) \le u) \quad and \quad F^U(u) = Pr(\theta \in [0,1] : U(\theta) \le u). \tag{41}$$

The key observation is now that the second part of (39) and (40) imply that $F^{\tilde{U}}$ is a mean preserving spread of F^U. Therefore, because Π is concave, the third term in (38) is larger for U than for $\tilde{U}$. □

For the general case without uniform distribution, the construction is analogous. The mean preserving spread argument carries over unchanged. The role of the regularity condition is to sign what corresponds to the first and second terms in (38), since these terms depend in general on the density f.

The regularity condition in Proposition 1 is not entirely new to the literature. In a context where the principal is a seller and the agent is a buyer, Manelli and Vincent (2006, Theorem 4) impose an equivalent regularity condition when characterizing the profit maximizing solution in a multi-dimensional screening problem. A sufficient condition for the regularity condition is that jointly $f' \leq 0$ and f is log-convex.[19] Examples include the uniform distribution of our example, and, more generally the family of power distributions $F(\theta) = \theta^\alpha$, $\theta \in [0,1]$, for $\alpha \leq 1$, or the family of exponential distributions $F(\theta) = 1 - e^{-\lambda\theta}$, $\theta \geq 0$, $\lambda \geq 0$.

While our regularity condition is more restrictive than other regularity condition often found in mechanism design (such as monotone hazard rates), we stress that the condition is not a tight, but only a sufficient condition that ensures the optimality of a cutoff-contract and thus a deterministic allocation in period 1. In general, if our regularity condition is violated, solving R_1 becomes complicated for two reasons: first, the incentive constraints do not rule out non-monotone allocations. Second, even if one could show that a monotone allocation is optimal, the principal's problem is not a linear problem, and the trade-off between period 1 profits (which are linear in U) and period 2 profits (which are concave in U) does generally lead to "interior" solutions that do not correspond to allocations with possibly multiple cutoffs.

As mentioned above, the regularity condition is needed to control the sign of the principal's first period profit. Therefore, our results go through without the regularity condition whenever second period profits are sufficiently higher than first period profits, for example, when the period 2 project has a much larger scale than the period 1 project.[20]

[19] To see this, note

$$\frac{d}{d\theta}(v-\theta)\frac{f'(\theta)}{f(\theta)} = -\frac{f'(\theta)}{f(\theta)} + (v-\theta)\frac{d}{d\theta}\frac{f'(\theta)}{f(\theta)} = -\frac{f'(\theta)}{f(\theta)} + (v-\theta)\frac{d}{d\theta}\log(f(\theta)). \tag{42}$$

Because $v-\theta$ is positive on the range $[\underline{\theta}, \min\{v, \bar{\theta}\}]$, this expression is postive if $f' \leq 0$ and $\log f$ is increasing, that is, f is log-convex.

[20] More precisely, scale up the second period by multiplying the valuation v_2 and costs θ_2 by a factor $\lambda \geq 1$ so that the principal's continuation profit Π is increasing in λ. As shown in (99) in the appendix, the difference between the principal's profit from an arbitrary and a cutoff contract becomes $\Delta W = \Delta W_1 + \Delta W_2$ with $\Delta W_1 = \int_{\underline{\theta}}^{v}[(v-\theta)\frac{f'(\theta)}{f(\theta)} - 1][\tilde{U}(\theta) - \hat{U}(\theta)]\, dF(\theta)$ and $\Delta W_2 = \int_{\underline{\theta}}^{\bar{\theta}} \Pi(\hat{U}(\theta)) - \Pi(\tilde{U}(\theta))\, dF(\theta)$. Our stochastic dominance argument implies that for any distribution, we have $\Delta W_2 > 0$. The regularity condition implies that $\Delta W_1 > 0$ so that in this case, ΔW is always for any λ. If the regularity conditions fails, then we have $\Delta W_2 > 0$ but may have $\Delta W_1 < 0$. In this case, ΔW is positive when $\Delta W_2 > 0$ is sufficiently large, which is the case if λ is sufficiently large.

Proposition 1 shows that a cutoff-contract is a solution to the relaxed problem R_1. It is straightforward to verify that any cutoff-contract satisfies the constraints IC of the original problem. Therefore, we have:

Proposition 2 *Let $(v-\theta)\frac{f'(\theta)}{f(\theta)}$ be increasing on the range $[\underline{\theta}, \min\{v, \bar{\theta}\}]$, then there is a cutoff-contract $(x, U) \in \Lambda$ which solves problem P_1'. Moreover, a cutoff contract also satisfies constraint IC^0. Thus, it is a solution to the original problem P.*

Since a cutoff-contract consists only of the two parameters θ_0, U_0, finding the optimal cutoff-contract comes down to solving an optimization problem in two variables. We illustrate this exercise in our running example.

Example: For our uniform example, the principal's objective is

$$W(\theta_0, U_0) = \int_0^{\theta_0} 1-\theta \, d\theta + \int_0^{\theta_0} \Pi(U_0-\theta)\, d\theta + \int_{\theta_0}^1 \Pi(U_0-\theta_0)\, d\theta \qquad (43)$$

with $U_0 \geq \theta_0$ and where Π is given by (28). To determine the maximizer, we first determine an optimal $\theta_0^*(U_0)$ for a given U_0. A tedious but otherwise straightforward analysis of the first and second order condition with respect to θ_0 yields[21]

$$\theta_0^*(U_0) = U_0 - 1/18. \qquad (46)$$

Next, we maximize $W(\theta_0^*(U_0), U_0) = W(U_0-1/18, U_0)$ with respect to U_0. For $U_0 \leq 1/2$, this expression reduces to $109/648+(5+4\sqrt{2U_0}-9U_0)U_0/6$ which is strictly increasing for $U_0 \leq 1/2$ so that a maximum exhibits $U_0 \geq 1/2$. For $U_0 > 1/2$, the expression $W(U_0-1/18, U_0)$ reduces to the quadratic expression $41/324+4/3\cdot U_0 - U_0^2$ which attains a maximum at $U_0 = 2/3$.

We therefore conclude that $(\theta_0^*, U_0^*) = (11/18, 2/3)$ maximizes $W(\theta_0, U_0)$ with a payoff of

[21] The first order condition with respect to θ_0 is:

$$\frac{\partial W}{\partial \theta_0} = (1-\theta_0)(1-\Pi'(U_0-\theta_0)) = 0 \quad \Leftrightarrow \quad \theta_0 = 1 \text{ or } \Pi'(U_0-\theta_0) = 1. \qquad (44)$$

It is easy to check that $\theta_0 = 1$ is not a maximizer of W. By (28), the unique solution to $\Pi'(U_0-\theta_0) = 1$ is $\theta_0 = U_0 - 1/18$. This is indeed a maximizer of $W(\theta_0, U_0)$, because the second order condition is

$$\frac{\partial^2 W}{\partial \theta_0^2} = -1 + \Pi'(U_0-\theta_0) + \Pi''(U_0-\theta_0)(1-\theta_0) < 0, \qquad (45)$$

is satisfied for $\theta_0 = U_0 - 1/18$, since the first two terms cancel, while $\Pi''(U) = -\sqrt{2}U^{-3/2}/4 < 0$ for $U \leq 1/2$ and $U_0 - \theta_0 = 1/18 < 1/2$.

	FB	sSB	No LC	LC
P's payoff	1	.5	.75	.571
U's payoff	0	.25	.125	.242
Surplus	1	.75	.875	.813

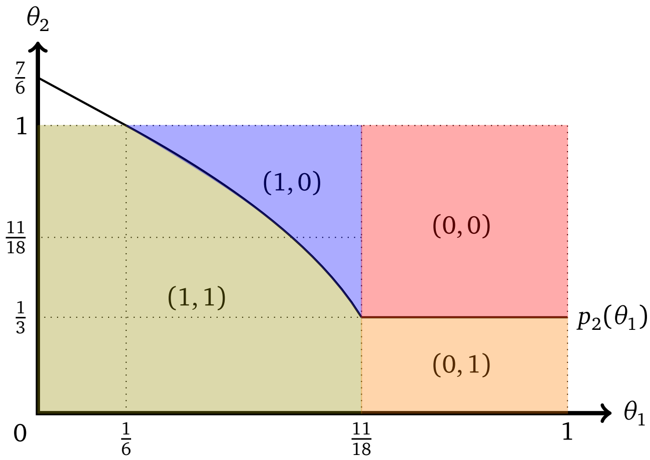

Figure 2: The table in the left panel presents the payoff comparisons of the four cases i) first best (FB); ii) static second best (sSB); iii) no liquidity constraints (no LC); and iv) liquidity constraints (LC). The graph in the right panel displays how optimal production (x_1, x_2) over the two periods depend on type combinations (θ_1, θ_2), splitting the type space $[0,1] \times [0,1]$ into four regions, where the curve $p_2(\theta_1)$ is the price that is paid to the agent for production in period 2 after having reported θ_1 in period 1.

$185/324 \approx 0.571$, exceeding by 14% the principal's payoff of $2\Pi^{SB} = 1/2$, from charging twice the static optimal price $p = 1/2$.

Recall from above that in the uniform example, the period 2 terms of trade can be implemented by offering the agent a period 2 price $p_2 = \sqrt{2U}$ for $U \leq 1/2$, and $p_2 = U + 1/2$ for $U > 1/2$. With this in mind, period 1 cost types θ above the cutoff $\theta_0^* = 11/18$ do not produce in period 1 and obtain expected period 2 utility of $U(\theta) = U_0^* - \theta_0^* = 2/3 - 11/18 = 1/18$ corresponding to a period 2 price offer $p_2 = 1/3$. All period 1 cost types θ below the cutoff $\theta_0^* = 11/18$ produce in the first period and obtain expected period 2 utility $U(\theta) = U_0^* - \theta = 2/3 - \theta$. This corresponds to a period 2 price offer $p_2(\theta_1) = U(\theta_1) + 1/2 = 7/6 - \theta_1$ for $\theta_1 \in [0, 1/6)$, and $p_2(\theta_1) = \sqrt{2U(\theta_1)} = \sqrt{4/3 - 2\theta_1}$ for $\theta_1 \in [1/6, 11/18)$. Interestingly, period 1 cost types $\theta < 1/6$ obtain a continuation utility U larger than $1/2$. These types always produce in period 2, since they receive the offer to produce at a price larger than 1 in period 2. In particular, the principal suffers a period 2 loss in this case.

The ex ante expected utility of the agent is $157/648$ so that expected aggregate surplus is $185/324 + 157/648 = 527/648 \approx 0.813$, compared to the first best surplus of 1. Without liquidity constraints, aggregate surplus is 7/8=.875, while the twicely repeated static second best contract yields aggregate surplus of .75. The table in the left panel of Figure 2 summarizes payoffs for the different types of models. The graph in the right panel displays the dependence of optimal production over the two periods on the first and second period type combinations (θ_1, θ_2). At our optimum, period 1 types produce if and only if $\theta_1 \leq 11/18$, whereas period 2 types produce if and only if $\theta_2 \leq p_2(\theta_1)$, i.e., when their costs lies below the price $p_2(\theta_1)$. As the graph illustrates, this yields four different regions. While the aggregate trade volume slightly increases from 11/18 in period 1 to 50/81 in period 2[22], we point out that conditional on cost type, trade does not become more efficient: for example, the first period type $\theta_1 = 11/18 - \epsilon$ always trades, but the same second period type $\theta_2 = 11/18 - \epsilon$ does not. In this sense, it is not clear-cut in which of the periods the allocation is more efficient. □

Remark 2 (Implementation) We now briefly discuss how an optimal cutoff contract can be indirectly implemented by a menu of prices. For simplicity, suppose that the optimal period 2 terms of trade can be implemented by a posted price. Recall from Remark 1 that this is the case if, for example, F_2/f_2 is increasing.

An optimal contract can then be implemented by a menu $\{(r, p_2(r)) \mid r \in [\underline{\theta}, \theta_0]\}$ where the agent can choose to produce the good in period 1 for a price r and conditional on not going bankrupt in period 1, obtains the option to produce the good in period 2 for the price $p_2(r)$ where p_2 is decreasing in r. Moreover, if the agent goes bankrupt in period 1, the relationship is terminated.

To see this, recall that under a backloaded contract, the agent breaks even in period 1. Under a cutoff contract, the agent therefore receives in period 1 the transfer $\hat{\theta}_1$ and produces the good if he announces $\hat{\theta}_1 \in [\underline{\theta}, \theta_0]$ and stays liquid. If he announces $\hat{\theta}_1 \in (\theta_0, \bar{\theta}]$, he receives the transfer 0 and does not produce the good. This corresponds to choosing a price $r = \hat{\theta}_1 \in [\underline{\theta}, \theta_0]$ at which to deliver the good in period 1. Moreover, after announcing $\hat{\theta}_1$, the agent obtains expected utility $U(\hat{\theta}_1)$ in period 2 which can be implemented by a posted price $p_2(\hat{\theta}_1)$ which is decreasing in $\hat{\theta}_1$ because $U(\hat{\theta}_1)$ is decreasing in $\hat{\theta}_1$. This corresponds to obtaining the option to produce the good at $p_2(r) = p_2(\hat{\theta}_1)$ in period 2 after choosing the price r in period 1.

Remark 3 (Finitely many types) A novelty of our paper is that we study continuous types. Ashlagi et al. (2023), like us, consider a linear, unit-good framework but with finitely many types and where trade is always efficient. Their analysis shows that the case with more than two finitely many types is analytically intractable. When there are two types, any contract that is feasible (with their liquidity constraint) is a cut-off contract by definition, and their Proposition 4 shows that a variety of deterministic contracts can be optimal, among them non-dynamic posted-price contracts where the second period posted-price is independent of the first period report. Our

[22]The trade volume in period 2 is $1/6 + \int_{1/6}^{11/18} \sqrt{4/3 - 2\theta_1}\, d\theta_1 + 7/18 \times 1/3$.

analysis suggests that this is a special feature of the two types case since with continuous types the second period posted price always depends on the first period report (see the previous remark).

Remark 4 (Off-path liquidity constraints) As explained before Section 4, the approach of the existing literature differs from our approach in that it considers the bi-directional version of IC_1 and not having IC_0^1 in problem P. Therefore, in the recursive formulation, the period 2 problem is the same under both approaches. Moreover, since we consider a relaxed version without IC_0^1, the period 1 problem the literature considers is precisely problem P_1 with the difference that IC_1 is replaced by its bi-directional counterpart. Now, the solution to P_1 exhibits the revenue equivalence property $U'(\theta) = -x(\theta)$ (by Lemma 5) and, as a cutoff contract, displays a monotone allocation. Therefore, the solution indeed satisfies the bi-directional counterpart of IC_1 and is thus also a solution to the problem studied in the literature.

When considering the bi-directional version of IC_1, however, the following inconsistency arises: consider a high cost type $\theta > \theta_0$ (who does not produce the good) and a low cost type $\hat{\theta} < \theta_0$ (who does produce the good). Type θ's continuation value is lower than that of type $\hat{\theta}$: $U(\theta) < U(\hat{\theta})$. Imposing the bi-directional version of IC_1 implies that type θ could obtain this higher continuation value by reporting $\hat{\theta}$. Therefore, what ensures bi-directional incentive compatibility is that one needs to assume that type θ would suffer a period 1 loss of $t(\hat{\theta}) - \theta < 0$ from reporting to be type $\hat{\theta}$. Clearly, this is inconsistent with the agent being periodically liquidity constraint. In our approach, this inconsistency does not arise, because what ensures incentive compatibility is that type θ would become illiquid when reporting to be type $\hat{\theta}$, and then, as this is verifiable, obtain a continuation value of zero.

Remark 5 (More than two periods) While we performed our analysis only for two periods, the extension to multiple periods is straightforward. To illustrate, suppose that there are infinitely many periods and that cost types θ_τ are i.i.d. with time-independent cdf F on the support $[\underline{\theta}, \bar{\theta}]$.[23] For the problem to be well-defined, assume that both parties discount future payoffs with a discount factor $\delta \in [0, 1)$. Under the dynamic programming formulation, the principal's choice variables are a probability of trade $x(\theta)$ for the current period and the expected continuation utility for the agent $U(\theta)$ that both depend on a report θ by the agent about his current type (as well as on the history of past reports which we suppress). The principal's value function $\Pi(V)$ is now defined recursively as a function of the agent's expected utility V (starting as of now)

[23] The extension to an arbitrary finite time horizon is analogous but all expressions are time-dependent.

according to the dynamic program:[24]

$$
\begin{aligned}
P_\infty: \quad \Pi(V) = \max_{x,U} \quad & \int_{\underline{\theta}}^{\bar{\theta}} (v-\theta)x(\theta) + \delta\Pi(U(\theta))\, dF(\theta) \quad s.t. && (47)\\
IR: \quad & U(\theta) \geq 0 \quad \forall\theta && (48)\\
IC: \quad & U(\theta) \geq (\hat{\theta}-\theta)x(\hat{\theta}) + U(\hat{\theta}) \quad \forall\theta \leq \hat{\theta} && (49)\\
UG: \quad & x(\theta) \in [0,1] \quad \forall\theta && (50)\\
PK: \quad & \int_{\underline{\theta}}^{\bar{\theta}} \delta U(\theta) dF(\theta) = V. && (51)
\end{aligned}
$$

While problem P_∞ yields the principal's value function, the solution to the principal's overall problem starting in the initial period is obtained by maximizing Π with respect to V.

The essential difference between P_∞ and P_1' is the presence of the promise keeping constraint PK which ensures that the agent's expected utility from the contract is V. As above, we consider a relaxed problem where we localize IC and replace it with the constraints M and IC_L as stated in Lemma 4:

$$
R_\infty: \quad \tilde{\Pi}(V) = \max_{x,U} \int_{\underline{\theta}}^{\bar{\theta}} (v-\theta)x(\theta) + \delta\tilde{\Pi}(U(\theta))\, dF(\theta) \quad s.t \quad IR, M, IC_L, UG, PK. \tag{52}
$$

It follows from standard arguments (see Stockey and Lucas, 1989, or Krishna et al. 2013) that $\tilde{\Pi}$ exists. Crucially, as in the two-period case, $\tilde{\Pi}$ is concave. Recall that to establish the optimality of a cutoff contract for the two-period problem R_1, we exploited the concavity of $\tilde{\Pi}$ to construct for a every feasible contract $(\tilde{x}, \tilde{U})$ a feasible cutoff-contract (x, U) that is an improvement. Note that, in contrast to problem R_1, feasibility in problem R_∞ requires that a contract, in addition, satisfies PK. Therefore, to extend the argument from R_1 to R_∞, we have to ensure that the cutoff contract (x, U) that improves a given feasible contract does satisfy PK.

However, note that the cutoff contract (x, U) constructed in the two-period problem to improve upon $(\tilde{x}, \tilde{U})$ has the property that[25]

$$
\int_{\underline{\theta}}^{\bar{\theta}} U(\theta) dF(\theta) = \int_{\underline{\theta}}^{\bar{\theta}} \tilde{U}(\theta) dF(\theta). \tag{53}
$$

[24]The formulation implicitly assumes that contracts are backloaded and excess payments are paid out "at infinity". This assumption is standard in the literature, and one motivation of it is that the discount factor corresponds to the probability that the relationship does not terminate in the next round, and excess payments are made after termination (which happens in finite time with probability 1).

[25]This corresponds to the right part of (39) where we defined (x, U) in the uniform example.

Therefore, as $(\tilde{x}, \tilde{U})$ is an arbitrary feasible contract and thus satisfies PK by definition, so does (x, U). This shows that a cutoff contract is optimal also when there are more than two periods.

Remark 6 (Portability) We have solved for an optimal contract using a new method that ranks contracts in terms of the spread of the distribution of the induced continuation values for the agent. An open question is to what extent this method can be employed in a model with correlated cost types (as in Krasikov and Lamba, 2021). Such an extension is beyond the scope of the current paper because it implies that the agent's continuation value becomes type dependent, thus constraining the principal's choice of continuation values.

Outside of the context of this paper, our method is applicable to static mechanism design problems with standard bilateral incentive constraints where agents have linear utility functions and the principal's payoff is concave in the agent's information rent. An example is optimal redistribution by a social planner who assigns (after-tax) payments t and (pre-tax) "labour income" x to each of a continuum of agents' who each privately know their labour cost θ. The social planner seeks to maximize a social welfare function $\int \Pi(U(\theta))\, dF(\theta)$ subject to the budget constrained that after-tax payments are lower than pre-tax income in the aggregate. The concavity of Π captures the planner's redistribution concerns. Our solution method is applicable when agents' preferences are linear in type and transfer. In this case, the (slope of) the indirect utility U is a function of the allocation, and analogous steps as above can be used to write the planner's problem as a constrained maximization problem over the indirect utility function where the (Langrangian) objective ranks indirect utility functions depending on how spread out they are. We leave a detailed analysis for future research.

5 Conclusion

We study short-term liquidity constraints in an otherwise standard dynamic screening model. We argue that modelling a liquidity constraint as a hard, physical constraint whose violation forces the agent to renege on the current contract terms implies that such a violation must be seen as a verifiable event on which a long-term contract can condition . We show how this yields a consistent framework in which liquidity concerns affect contractual feasibility constraints by giving rise to one-sided incentive compatibility constraints.

While our assumption that the agent runs into liquidity problems results whenever the agent makes short term losses is in line with standard approaches, in practice the occurrence and consequences of liquidity problems may be more complicated than that, since they may, for example, be partially discretionary or involve restructuring processes which would affect the extent to which liquidity problems are verifiable. It is an interesting avenue for future research to capture such richer forms of liquidity concerns.

Appendix

Proof of Lemma 1 Let $\tilde{\gamma} = (\tilde{x}_1, \tilde{t}_1, \tilde{x}_2^L, \tilde{t}_2^L, \tilde{x}_2^I, \tilde{t}_2^I)$ be a feasible contract. Our proof strategy is to first define an auxiliary contract $\hat{\gamma}$ that is feasible and payoff-equivalent to $\tilde{\gamma}$ but under which the agent never becomes illiquid. In a second step, we modify $\hat{\gamma}$ to obtain the desired contract γ that has the properties stated in the lemma. In what follows, we indicate all variables pertaining to $\tilde{\gamma}$ and $\hat{\gamma}$ with a tilde and a hat.

Step 1: Define the auxiliary contract $\hat{\gamma} = (\hat{x}_1, \hat{t}_1, \hat{x}_2^L, \hat{t}_2^L, \hat{x}_2^I, \hat{t}_2^I)$ by

$$(\hat{x}_1(\theta_1), \hat{t}_1(\theta_1)) = \begin{cases} (\tilde{x}_1(\theta_1), \tilde{t}_1(\theta_1)) & \text{if } \theta_1 \in \tilde{\Theta}_1^L, \\ (0,0) & \text{otherwise} \end{cases} \quad (54)$$

$$(\hat{x}_2^L(\theta_1,\theta_2), \hat{t}_2^L(\theta_1,\theta_2)) = \begin{cases} (\tilde{x}_2^L(\theta_1,\theta_2), \tilde{t}_2^L(\theta_1,\theta_2)) & \text{if } \theta_1 \in \tilde{\Theta}_1^L, \theta_2 \in \tilde{\Theta}_2^{L,L}(\theta_1) \\ (\tilde{x}_2^I(\theta_1,\theta_2), \tilde{t}_2^I(\theta_1,\theta_2)) & \text{if } \theta_1 \notin \tilde{\Theta}_1^L, \theta_2 \in \tilde{\Theta}_2^{I,L}(\theta_1) \\ (0,0) & \text{otherwise,} \end{cases} \quad (55)$$

$$(\hat{x}_2^I(\theta_1,\theta_2), \hat{t}_2^I(\theta_1,\theta_2)) = (0,0) \quad \forall \theta_1, \theta_2. \quad (56)$$

We show that $\hat{\gamma}$ is feasible and payoff-equivalent to $\tilde{\gamma}$. To see this, note first that, by construction, we have $\hat{\Theta}_1^L = \Theta_1$ and $\hat{\Theta}_2^{L,L}(\theta_1) = \Theta_2$ for all θ_1. Furthermore,

$$\hat{U}^L(\theta_1) = \tilde{U}^L(\theta_1) \text{ for } \theta_1 \in \tilde{\Theta}_1^L \quad \textit{and} \quad \hat{U}^L(\theta_1) = \tilde{U}^I(\theta_1) \text{ for } \theta_1 \notin \tilde{\Theta}_1^L. \quad (57)$$

To see feasibility, observe that $\hat{\gamma}$ trivially satisfies (4) for $\ell = I$, and inherits (4) for $\ell = L$ by construction. To see (5), let $\hat{t}_1(\hat{\theta}_1) - \theta_1 \hat{x}_1(\hat{\theta}_1) \geq 0$. Consider first the case that $\theta_1 \in \tilde{\Theta}_1^L$ and $\hat{\theta}_1 \in \tilde{\Theta}_1^L$. Then, we have:

$$\begin{aligned} \hat{t}_1(\theta_1) - \theta_1 \hat{x}_1(\theta_1) + \hat{U}^L(\theta_1) &= \tilde{t}_1(\theta_1) - \theta_1 \tilde{x}_1(\theta_1) + \tilde{U}^L(\theta_1) && (58) \\ &\geq \tilde{t}_1(\hat{\theta}_1) - \theta_1 \tilde{x}_1(\hat{\theta}_1) + \tilde{U}^L(\hat{\theta}_1) && (59) \\ &= \hat{t}_1(\hat{\theta}_1) - \theta_1 \hat{x}_1(\hat{\theta}_1) + \hat{U}^L(\hat{\theta}_1), && (60) \end{aligned}$$

where the inequality follows, because $\tilde{\gamma}$ satisfies (5) and the two equalities follow from (57). The other cases can be shown analogously.

To see (6), note that the left hand side of (6) is non-negative by definition of $\hat{\gamma}$. Moreover, because $\hat{x}_2^I = \hat{t}_2^I = 0$, we have $\hat{U}^I(\hat{\theta}_1) = 0$ for all $\hat{\theta}_1$ so that the right hand side is zero. Therefore, (6) follows. To complete the proof of feasibility, note that (7) and (8) are void for $\hat{\gamma}$, because

$\hat{\Theta}_1^L = \Theta_1$.

Finally, $\hat{\gamma}$ and $\tilde{\gamma}$ are payoff-equivalent, because by construction, if the agent is liquid under $\tilde{\gamma}$, then $\hat{\gamma}$ implements the same terms of trade as $\tilde{\gamma}$, and when the agent becomes illiquid under $\tilde{\gamma}$, no trade occurs under $\hat{\gamma}$ so that under either contract both the principal and the agent get zero.

Step 2: We now construct a feasible contract $\gamma = (x_1, t_1, x_2^L, t_2^L, x_2^I, t_2^I)$ which is payoff-equivalent to $\hat{\gamma}$ and satisfies (12)-(14). To do so, note first that $\hat{\gamma}$ satisfies (13) and (14), but may violate (12) and display $\hat{t}_1(\theta_1) - \theta_1 \hat{x}_1(\theta_1) > 0$ for some θ_1.

Define γ as the contract that differs from $\hat{\gamma}$ only in that the period 1 profits for the agent are backloaded to period 2. Formally, γ displays $x_1 = \hat{x}_1, x_2^L = \hat{x}_2^L, x_2^I = \hat{x}_2^I, t_2^I = \hat{t}_2^I$ and payments

$$t_1(\theta) = \theta_1 x_1(\theta_1), \quad t_2^L(\theta_1, \theta_2) = \hat{t}_2^L(\theta_1, \theta_2) + \hat{t}_1(\theta_1) - t_1(\theta_1). \tag{61}$$

Note first that γ satisfies (12) by construction. Moreover, it inherits (14) from $\hat{\gamma}$ and also property (13) because

$$t_2^L(\theta_1, \theta_2) - \theta_2 x_2^L(\theta_1, \theta_2) = \hat{t}_2^L(\theta_1, \theta_2) + \hat{t}_1(\theta_1) - t_1(\theta_1) - \theta_2 \hat{x}_2^L(\theta_1, \theta_2) \tag{62}$$

$$= \hat{t}_2^L(\theta_1, \theta_2) - \theta_2 \hat{x}_2^L(\theta_1, \theta_2) + \hat{t}_1(\theta_1) - \theta_1 \hat{x}_1(\theta_1) \geq 0, \tag{63}$$

where the inequality follows since under $\hat{\gamma}$

dan, the agent is never illiquid.

We next show that γ is feasible. Indeed, γ trivially satisfies (4) for $\ell = I$ because $x_2^I = t_2^I = 0$. For $\ell = L$, we have for all $\theta_1, \theta_2, \hat{\theta}_2$:

$$t_2^L(\theta_1, \theta_2) - \theta_2 x_2^L(\theta_1, \theta_2) = \hat{t}_2^L(\theta_1, \theta_2) + \hat{t}_1(\theta_1) - t_1(\theta_1) - \theta_2 x_2^L(\theta_1, \theta_2) \tag{64}$$

$$\geq \hat{t}_2^L(\theta_1, \hat{\theta}_2) + \hat{t}_1(\theta_1) - t_1(\theta_1) - \theta_2 x_2^L(\theta_1, \hat{\theta}_2) \tag{65}$$

$$= t_2^L(\theta_1, \hat{\theta}_2) - \theta_2 x_2^L(\theta_1, \hat{\theta}_2), \tag{66}$$

where the first and the third lines use the definition of t_2^L, and the second line follows because $\hat{\gamma}$ satisfies (4) for $\ell = L$ and since $x_2^L = \hat{x}_2^L$.

To see (5), consider $\theta_1, \hat{\theta}_1$ so that $t_1(\hat{\theta}_1) - \theta_1 x_1(\hat{\theta}_1) \geq 0$. Because $\hat{t}_1(\hat{\theta}_1) \geq t_1(\hat{\theta}_1)$ and $\hat{x}_1(\hat{\theta}) = x_1(\hat{\theta})$, this implies that also $\hat{t}_1(\hat{\theta}_1) - \theta_1 \hat{x}_1(\hat{\theta}_1) \geq 0$. Therefore, since $\hat{\gamma}$ satisfies (5), we have

$$\hat{t}_1(\theta) - \theta_1 \hat{x}_1(\theta_1) + \hat{U}^L(\theta_1) \geq \hat{t}_1(\hat{\theta}) - \theta_1 \hat{x}_1(\hat{\theta}_1) + \hat{U}^L(\hat{\theta}_1). \tag{67}$$

Moreover, by construction, we have that $t_1(\theta_1)+U^L(\theta_1) = \hat{t}_1(\theta_1)+\hat{U}^L(\theta_1)$. These two observations imply that

$$
\begin{aligned}
t_1(\theta_1) - \theta_1 x_1(\theta_1) + U^L(\theta_1) &= \hat{t}_1(\theta) - \theta_1 \hat{x}_1(\theta_1) + \hat{U}^L(\theta_1) && (68)\\
&\geq \hat{t}_1(\hat{\theta}) - \theta_1 \hat{x}_1(\hat{\theta}_1) + \hat{U}^L(\hat{\theta}_1) && (69)\\
&= t_1(\hat{\theta}) - \theta_1 x_1(\hat{\theta}_1) + U^L(\hat{\theta}_1). && (70)
\end{aligned}
$$

Furthermore, γ satisfies (6), because $U^I(\hat{\theta}_1) = 0$ for all $\hat{\theta}_1$ and the left hand side of (6) is non-negative. Finally, (7) and (8) are void for γ, because $\Theta_1^L = \hat{\Theta}_1^L = \Theta_1$.

It remains to show that γ and $\hat{\gamma}$ are payoff-equivalent. But this follows, because the only difference between the contracts is that the payments have been moved between periods, but the sum of payments over the two periods is the same. qed

Proof of Lemma 2 Let $\gamma = (x_1, t_1, x_2^L, t_2^L, x_2^I, t_2^I)$ be the contract induced by the backloaded contract (x_1, x_2, t_2). Hence, $t_1 = \theta_1 x_1$, $x_2^L = x_2$,$t_2^L = t_2$, $x_2^I = t_2^I = 0$. We have to show that γ is feasible if and only if IC_2 and IC_1 hold. To see this, observe first that γ trivially satisfies (4) for $\ell = I$ because $x_2^I = t_2^I = 0$. Moreover, for any backloaded-induced contract γ, the constraint (4) for $\ell = L$ rewrites as IC_2. Hence γ satisfies (4) if and only if it satisfies IC_2.

We next show that constraint (5) is equivalent to IC_1. Indeed, since $t_1(\hat{\theta}_1) = \hat{\theta}_1 x_1(\hat{\theta}_1)$ for all $\hat{\theta}_1$, we have

$$
t_1(\hat{\theta}_1) - \theta_1 x_1(\hat{\theta}_1) \geq 0 \quad \Leftrightarrow \quad (\hat{\theta}_1 - \theta_1) x_1(\hat{\theta}_1) \geq 0 \quad \Leftrightarrow \quad \theta_1 \leq \hat{\theta}_1. \tag{71}
$$

Hence, γ satisfies (5) if and only if for all $\theta_1 \leq \hat{\theta}_1$, we have $t_1(\theta_1) - \theta_1 x_1(\theta_1) + U(\theta_1) \geq t_1(\hat{\theta}_1) - \theta_1 x_1(\hat{\theta}_1) + U(\hat{\theta}_1)$. But because $t_1(\theta_1') - \theta_1' x_1(\theta_1') = 0$ for all θ_1' holds for any contract γ that is induced by some backloaded contract, this is equivalent to IC_1.

Moreover, γ always satisfies (6) because the right hand side of (6) is zero, and the left hand side is non-negative. Finally, (7) and (8) are void for γ because $\Theta_1^L = \Theta_1$. This completes the proof. qed

Proof of Lemma 3 To simplify notation, we omit θ_1 and suppress the time subindex. With standard screening arguments, we can write P_2 as a maximization problem that maximizes the virtual surplus with respect to the allocation $x(\cdot)$ and the rent of the most inefficient type $u(\bar{\theta})$ as follows:

$$P_2: \qquad \Pi(U) \equiv \max_{x,u(\bar{\theta})} \quad \int_{\underline{\theta}}^{\bar{\theta}} \left(v - \theta - \frac{F(\theta)}{f(\theta)}\right) x(\theta)\, dF(\theta) - u(\bar{\theta}) \quad s.t \tag{72}$$

$$M: \quad x(\theta) \text{ is decreasing in } \theta \tag{73}$$

$$L_2: \quad u(\bar{\theta}) \geq 0 \tag{74}$$

$$PK: \quad u(\bar{\theta}) + \int_{\underline{\theta}}^{\bar{\theta}} x(\theta) \frac{F(\theta)}{f(\theta)}\, dF(\theta) = U \tag{75}$$

That $\Pi(U)$ attains a maximum Π_2^{SB} at U_2^{SB} is explained in the main text.

To see the further claims of the Lemma note that PK pins down $u(\bar{\theta})$, and by substituting out $u(\bar{\theta})$ in the objective (73) and L_2, the problem simplifies to

$$\hat{P}_2: \qquad \Pi(U) \equiv \max_{x} \quad \int_{\underline{\theta}}^{\bar{\theta}} (v - \theta) x(\theta)\, dF(\theta) - U \quad s.t \tag{76}$$

$$M: \quad x(\theta) \text{ is decreasing in } \theta \tag{77}$$

$$L_2: \quad \int_{\underline{\theta}}^{\bar{\theta}} x(\theta) \frac{F(\theta)}{f(\theta)}\, dF(\theta) \leq U \tag{78}$$

To see that $\Pi(0) = 0$, note that (78) implies that the only way to supply $U = 0$ is to have $x(\theta) = 0$ for all θ, resulting in zero profits, hence: $\Pi(0) = 0$.

To see that Π is concave, let x' resp. x'' be solutions to $\hat{P}_2$ for U' resp. U''. Then the allocation $\bar{x} = \alpha x' + (1-\alpha)x''$ satisfies M and L_2 for $U = \alpha U' + (1-\alpha)U''$. Moreover, $\bar{x}$ yields profit $\alpha\Pi(U') + (1-\alpha)\Pi(U'')$. The solution to $\hat{P}_2$ for $U = \alpha U' + (1-\alpha)U''$ must therefore yield at least $\bar{\Pi}$. Thus, we have $\Pi(\alpha U' + (1-\alpha)U'') \geq \alpha\Pi(U') + (1-\alpha)\Pi(U'')$, which establishes concavity of Π.

To see that $\Pi(S^{FB}) = 0$, note that, by definition, $\Pi + U \leq S^{FB}$ for any allocation $x(\cdot)$. Hence, we have $\Pi(S^{FB}) \leq 0$. To show $\Pi(S^{FB}) = 0$, it therefore suffices to show that, for $U = S^{FB}$, the first-best allocation $x^{FB}(\theta) = 1_{[\underline{\theta}, \min\{v,\bar{\theta}\}]}(\theta)$ satisfies (77) and (78) and yields 0 for the objective (76). Indeed, $x^{FB}(\theta)$ clearly satisfies (77) and, together with $U = S^{FB}$, yields 0 for the objective (76). To see that the first-best allocation also satisfies (78) for $U = S^{FB} \int_{\underline{\theta}}^{\min\{v,\bar{\theta}\}} v - \theta\; dF(\theta)$, note that by integration by parts:

$$\int_{\underline{\theta}}^{\bar{\theta}} x^{FB}(\theta)\frac{F(\theta)}{f(\theta)}\,dF(\theta) = \int_{\underline{\theta}}^{\min\{v,\bar{\theta}\}} \frac{F(\theta)}{f(\theta)}\,dF(\theta) \tag{79}$$

$$= -(v-\theta)F(\theta)\,\Big|_{\underline{\theta}}^{\min\{v,\bar{\theta}\}} + \int_{\underline{\theta}}^{\min\{v,\bar{\theta}\}} v-\theta\,dF(\theta) \le S^{FB}. \tag{80}$$

qed

Proof of Lemma 4 That U is decreasing is immediate from IC. Since U is decreasing, U has a derivative almost everywhere by Lebesque's Theorem. Now suppose that U' exists at θ. Note that for $h > 0$, we can write IC as $U(\theta - h) - U(\theta) \ge hx(\theta)$. Thus,

$$U'(\theta) = \lim_{h\to 0}\frac{U(\theta) - U(\theta - h)}{h} \le -x(\theta), \tag{81}$$

as desired. qed

Proof of Lemma 5 Let $(\tilde{x}, \tilde{U}) \in \Phi$ be such that it does not satisfy (i) or (ii). We construct an improvement $(x, U) \in \Phi$ that satisfies (i) and (ii).

Suppose that $v < \bar{\theta}$ and $(\tilde{x}, \tilde{U})$ violates (i). Consider first the case that $\tilde{U}(v) \le U^{SB}$, and define (x, U) as

$$x(\theta) = \begin{cases} \tilde{x}(\theta) & if \quad \theta \le v \\ 0 & if \quad \theta > v \end{cases}, \quad U(\theta) = \begin{cases} \tilde{U}(\theta) & if \quad \theta \le v \\ \tilde{U}(v) & if \quad \theta > v. \end{cases} \tag{82}$$

Clearly, $(x, U) \in \Phi$ and satisfies (i). We next argue that (x, U) is a (weak) improvement over $(\tilde{x}, \tilde{U})$ by showing that

$$\int_{\underline{\theta}}^{\bar{\theta}} (v-\theta)x(\theta)\,dF(\theta) \ge \int_{\underline{\theta}}^{\bar{\theta}} (v-\theta)\tilde{x}(\theta)\,dF(\theta), \text{ and} \tag{83}$$

$$\int_{\underline{\theta}}^{\bar{\theta}} \Pi(U(\theta))\,dF(\theta) \ge \int_{\underline{\theta}}^{\bar{\theta}} \Pi(\tilde{U}(\theta))\,dF(\theta). \tag{84}$$

Inequality (83) is immediate from the definition of x. To see (84), note that because U is decreasing and $U(v) \le U^{SB}$ by assumption, it follows by construction that for all $\theta > v$, we have $U^{SB} \ge U(\theta) \ge \tilde{U}(\theta)$. Thus, because Π is concave and uniquely maximized at U^{SB} by Lemma 3, this implies that $\Pi(U(\theta)) \ge \Pi(\tilde{U}(\theta))$ for all $\theta > v$. Since $U(\theta) = \tilde{U}(\theta)$ for all $\theta \le v$, (84) follows.

Next consider the case that $\tilde{U}(v) > U^{SB}$. Define (x,U) as

$$x(\theta)=\begin{cases}\tilde{x}(\theta) & if \quad \theta\leq v\\ 0 & if \quad \theta>v\end{cases}, \quad U(\theta)=\begin{cases}\tilde{U}(\theta) & if \quad \theta\leq v\\ U^{SB} & if \quad \theta>v.\end{cases} \tag{85}$$

Clearly, $(x,U)\in\Phi$ and satisfies (i). It follows with similar arguments as in the previous paragraph that (x,U) is a (weak) improvement over $(\tilde{x},\tilde{U})$.

Finally, suppose $(\tilde{x},\tilde{U})$ violates (ii). Define

$$\tau=\sup\{\theta \mid \tilde{U}(\theta)\geq U^{SB}\}. \tag{86}$$

Because U is decreasing, we have that

$$\tilde{U}(\theta)\geq U^{SB} \text{ for all } \theta<\tau, \quad \text{and} \quad \tilde{U}(\theta)<U^{SB} \text{ for all } \theta>\tau. \tag{87}$$

Define (x,U) as $x(\theta)=\tilde{x}(\theta)$ for all θ, and

$$U(\theta)=U^{SB}-\int_{\tau}^{\theta}x(t)\,dt. \tag{88}$$

Clearly, $(x,U)\in\Phi$ and satisfies (ii). To show that (x,U) yields a higher profit than $(\tilde{x},\tilde{U})$, observe that because $(\tilde{x},\tilde{U})$ and (x,U) specify the same allocation x, it is sufficient to show that

$$\Pi(\tilde{U}(\theta))\leq\Pi(U(\theta)) \text{ for almost all } \theta. \tag{89}$$

To see this, consider first the case that $\theta<\tau$. It is well-known that the derivative of a decreasing function is (Lebesgue) integrable and that $\tilde{U}(\theta)-\tilde{U}(\tilde{\theta})\geq\int_{\tilde{\theta}}^{\theta}\tilde{U}'(t)\,dt$ for all $\theta,\tilde{\theta}$. Hence, for all $\epsilon>0$ with $\theta<\tau-\epsilon$:

$$\begin{aligned}\tilde{U}(\theta) &\geq \int_{\tau-\epsilon}^{\theta}\tilde{U}'(t)\,dt+\tilde{U}(\tau-\epsilon) && (90)\\ &= -\int_{\theta}^{\tau-\epsilon}\tilde{U}'(t)\,dt+\tilde{U}(\tau-\epsilon) && (91)\\ &\geq \int_{\theta}^{\tau-\epsilon}\tilde{x}(t)\,dt+\tilde{U}(\tau-\epsilon) && (92)\\ &= -\int_{\tau-\epsilon}^{\theta}x(t)\,dt+\tilde{U}(\tau-\epsilon), && (93)\end{aligned}$$

where the second inequality follows from IC_L, and the final equality from $x = \tilde{x}$ Because the inequality holds for all $\epsilon > 0$ and since $\tilde{U}(\tau - \epsilon) \geq U^{SB}$ by (87), we can infer that

$$\tilde{U}(\theta) \quad \geq \quad -\int_{\tau}^{\theta} x(t)\,dt + U^{SB} = U(\theta). \tag{94}$$

Moreover, since $\theta < \tau$, we have $U(\theta) \geq U^{SB}$, and accordingly, $\tilde{U}(\theta) \geq U(\theta) \geq U^{SB}$. Because Π is concave and uniquely maximized at U^{SB} by Lemma 3, these inequalities imply (89) for $\theta < \tau$. A symmetrical argument works to show (89) for $\theta > \tau$, and this completes the proof. qed

Proof of Proposition 1 To avoid case distinctions, we only consider the case $v < \bar{\theta}$. By Lemma 5, it is sufficient to prove the statement for $(\tilde{x}, \tilde{U}) \in \Phi$ which satisfies properties (i) and (ii) from Lemma 5. Consequently, we have:

(i') $\tilde{U}(\theta) = \tilde{U}(v)$ for all $\theta \geq v$.

We first construct a contract $(\hat{x}, \hat{U})$ which is *not* necessarily in Λ that delivers a (weakly) more profit than $(\tilde{x}, \tilde{U})$. In a second step, we then construct (x, U) which *is* in Λ that delivers a (weakly) higher profit than $(\hat{x}, \hat{U})$.

As to the first step, define for $\alpha \in [\tilde{U}(v), \tilde{U}(\underline{\theta})]$ the two functions

$$\hat{U}_\alpha(\theta) = \begin{cases} \tilde{U}(\underline{\theta}) - (\theta - \underline{\theta}) & if \quad \theta \in [\underline{\theta}, \hat{\theta}] \\ \alpha & if \quad \theta \in (\hat{\theta}, v) \\ \tilde{U}(\theta) & if \quad \theta \in [v, \bar{\theta}] \end{cases}, \qquad \Delta(\alpha) \equiv \int_{\underline{\theta}}^{\bar{\theta}} \hat{U}_\alpha(\theta) - \tilde{U}(\theta)\,dF(\theta),$$

where $\hat{\theta} \equiv \underline{\theta} + \tilde{U}(\underline{\theta}) - \alpha \in [\underline{\theta}, v]$.

In words, $\hat{U}_\alpha$ starts at $\tilde{U}(\underline{\theta})$, then decreases with slope -1 until it attains the value α at the point $\hat{\theta}$, then stays constant equal to α until it reaches the point $\theta = v$, at which it jumps downwards to $\tilde{U}(v)$ and stays constant from then on (since it coincides with $\tilde{U}$ which is constant on $[v, \bar{\theta}]$ by (i') above)

Next, we show that there is $\hat{\alpha} \in [\tilde{U}(v), \tilde{U}(\underline{\theta})]$ so that

$$\int_{\underline{\theta}}^{\bar{\theta}} \hat{U}_{\hat{\alpha}}(\theta)\,dF(\theta) = \int_{\underline{\theta}}^{\bar{\theta}} \tilde{U}(\theta)\,dF(\theta). \tag{95}$$

Indeed, by construction, for $\alpha = \tilde{U}(\underline{\theta})$, we have $\hat{U}_\alpha(\theta) - \tilde{U}(\theta) \geq 0$ for all θ, and for $\alpha = \tilde{U}(v)$, we have $\hat{U}_\alpha(\theta) - \tilde{U}(\theta) \leq 0$ for all θ. It follows that $\Delta(\tilde{U}(\underline{\theta})) \geq 0$ and $\Delta(\tilde{U}(v)) \leq 0$. Because $\Delta(\alpha)$ is continuous on $\alpha \in [\tilde{U}(v), \tilde{U}(\underline{\theta})]$, the intermediate value theorem applies, implying (95).

Moreover, because $\hat{U}_{\hat{\alpha}}$ and $\tilde{U}$ coincide on $[v, \bar{\theta}]$ by construction, the previous equality can equivalently be written as

$$\int_{\underline{\theta}}^{v} \hat{U}_{\hat{\alpha}}(\theta)\, dF(\theta) = \int_{\underline{\theta}}^{v} \tilde{U}(\theta)\, dF(\theta). \tag{96}$$

From now on, denote $\hat{U}_{\hat{\alpha}}$ simply by $\hat{U}$. Moreover, let

$$\hat{x}(\theta) = \begin{cases} 1 & if \quad \theta \in [\underline{\theta}, \hat{\theta}] \\ 0 & if \quad \theta > \hat{\theta}. \end{cases} \tag{97}$$

We now show that $(\hat{x}, \hat{U})$ yields a (weakly) higher profit than $(\tilde{x}, \tilde{U})$. This is trivially the case for $\hat{\alpha} = \tilde{U}(v)$, where we have $(\hat{x}, \hat{U}) = (\tilde{x}, \tilde{U})$. Hence, suppose $\hat{\alpha} > \tilde{U}(v)$. In this case, we have $\hat{U}(v) > \tilde{U}(v)$. Therefore, because $\hat{U}(\underline{\theta}) = \tilde{U}(\underline{\theta})$, $\tilde{U}'(\theta) \geq \hat{U}'(\theta) = -1$ for $\theta \in [\underline{\theta}, \hat{\theta}]$ and $\tilde{U}'(\theta) \leq \hat{U}'(\theta) = 0$ for $\theta \in [\hat{\theta}, v]$, there is a $\tilde{\theta} \in [\underline{\theta}, v]$ so that

$$\hat{U}(\theta) - \tilde{U}(\theta) \leq 0 \quad \forall \theta \leq \tilde{\theta} \quad \text{and} \quad \hat{U}(\theta) - \tilde{U}(\theta) \geq 0 \quad \forall \theta \geq \tilde{\theta}. \tag{98}$$

Using the facts that $\hat{U}' = -\hat{x}$ and $\tilde{U}' = -\tilde{x}$, and $\hat{x}(\theta) = \tilde{x}(\theta) = 0$ for all $\theta > v$, we can write the difference in the principal's profits from $(\hat{x}, \hat{U})$ and $(\tilde{x}, \tilde{U})$ as

$$\begin{aligned} W(\hat{x}, \hat{U}) - W(\tilde{x}, \tilde{U}) &= \int_{\underline{\theta}}^{\bar{\theta}} (v - \theta)[\hat{x}(\theta) - \tilde{x}(\theta)] + \Pi(\hat{U}(\theta)) - \Pi(\tilde{U}(\theta))\, dF(\theta) \\ &= \int_{\underline{\theta}}^{v} (v - \theta)[\hat{x}(\theta) - \tilde{x}(\theta)]\, dF(\theta) + \int_{\underline{\theta}}^{\bar{\theta}} \Pi(\hat{U}(\theta)) - \Pi(\tilde{U}(\theta))\, dF(\theta) \\ &= \int_{\underline{\theta}}^{v} (v - \theta)[\tilde{U}'(\theta) - \hat{U}'(\theta)]\, dF(\theta) + \int_{\underline{\theta}}^{\bar{\theta}} \Pi(\hat{U}(\theta)) - \Pi(\tilde{U}(\theta))\, dF(\theta). \end{aligned}$$

Integrating the first integral by parts delivers

$$W(\hat{x}, \hat{U}) - W(\tilde{x}, \tilde{U}) = (v - \theta) f(\theta)[\tilde{U}(\theta) - \hat{U}(\theta)] \Big|_{\underline{\theta}}^{v} \tag{99}$$

$$- \int_{\underline{\theta}}^{v} [(v - \theta)\frac{f'(\theta)}{f(\theta)} - 1][\tilde{U}(\theta) - \hat{U}(\theta)]\, dF(\theta) \tag{100}$$

$$+ \int_{\underline{\theta}}^{\bar{\theta}} \Pi(\hat{U}(\theta)) - \Pi(\tilde{U}(\theta))\, dF(\theta). \tag{101}$$

We now argue that this expression is positive. Observe first that by construction, $\hat{U}(\underline{\theta}) = \tilde{U}(\underline{\theta})$, and thus the right hand side of (99) is equal to zero. Moreover, by (96), expression (100) can firstly be written as

$$-\int_{\underline{\theta}}^{v} [(v-\theta)\frac{f'(\theta)}{f(\theta)}][\tilde{U}(\theta)-\hat{U}(\theta)]\, dF(\theta) = Y, \tag{102}$$

and we can secondly add $\int_{\underline{\theta}}^{v}[(v-\tilde{\theta})\frac{f'(\tilde{\theta})}{f(\tilde{\theta})}][\tilde{U}(\theta)-\hat{U}(\theta)]\, dF(\theta) = 0$, with $\tilde{\theta}$ defined in (98), to obtain:

$$\begin{aligned} Y &= -\int_{\underline{\theta}}^{v} [(v-\theta)\frac{f'(\theta)}{f(\theta)} - (v-\tilde{\theta})\frac{f'(\tilde{\theta})}{f(\tilde{\theta})}][\tilde{U}(\theta)-\hat{U}(\theta)]\, dF(\theta) && (103)\\ &= -\int_{\underline{\theta}}^{\tilde{\theta}} [(v-\theta)\frac{f'(\theta)}{f(\theta)} - (v-\tilde{\theta})\frac{f'(\tilde{\theta})}{f(\tilde{\theta})}][\tilde{U}(\theta)-\hat{U}(\theta)]\, dF(\theta) && (104)\\ &\quad -\int_{\tilde{\theta}}^{v} [(v-\theta)\frac{f'(\theta)}{f(\theta)} - (v-\tilde{\theta})\frac{f'(\tilde{\theta})}{f(\tilde{\theta})}][\tilde{U}(\theta)-\hat{U}(\theta)]\, dF(\theta). && (105)\end{aligned}$$

Now, the assumption that $(v-\theta)\frac{f'(\theta)}{f(\theta)}$ is increasing implies that the first bracket under the integral (104) is negative for all $\theta \in [\underline{\theta}, \tilde{\theta}]$, and (98) implies that the second bracket under the integral (104) is positive for all $\theta \in [\underline{\theta}, \tilde{\theta}]$, so that, overall (104) is positive. Analogously, (105) is positive.

Finally, to see that (101) is positive, define for an arbitrary decreasing function U, the cdf F^U as the push-forward measure, that is, the utility distribution induced by U, given by

$$F^U(u) = Prob(\{\theta \mid U(\theta) \le u\}). \tag{106}$$

By (95) and (98), $F^{\tilde{U}}$ is a mean preserving spread of $F^{\hat{U}}$. Thus, because Π is concave by Lemma 3, we have

$$\int_{\underline{\theta}}^{\tilde{\theta}} \Pi(\hat{U}(\theta)) - \Pi(\tilde{U}(\theta))\, dF(\theta) = \int_{\underline{\theta}}^{\tilde{\theta}} \Pi(u)\, dF^{\hat{U}}(u) - \int \Pi(u)\, dF^{\tilde{U}}(u) \ge 0. \tag{107}$$

This completes the first step of the proof.

As to the second step, let $(\hat{x}, \hat{U})$ from the first step be given. We construct $(x, U) \in \Lambda$ which delivers a (weakly) more profit than $(\hat{x}, \hat{U})$. Indeed, let (x, U) be a cutoff-contract with cutoff

$\theta_0 = \hat{\theta}$ and an intercept $U_0 \in [\hat{U}(v) + \hat{\theta} - \underline{\theta}, \hat{U}(\underline{\theta})]$ such that[26]

$$\int_{\underline{\theta}}^{\bar{\theta}} U(\theta)\, dF(\theta) = \int_{\underline{\theta}}^{\bar{\theta}} \hat{U}(\theta)\, dF(\theta). \tag{108}$$

This also implies that

$$U(\theta) - \hat{U}(\theta) \leq 0 \quad \forall \theta \leq v \quad \text{ and } \quad U(\theta) - \hat{U}(\theta) \geq 0 \quad \forall \theta \geq v. \tag{109}$$

Because $\theta_0 = \hat{\theta}$ implies $x = \hat{x}$, the difference in the principal's profit from (x, U) and $(\hat{x}, \hat{U})$ can be written as

$$\begin{aligned} W(x,U) - W(\hat{x},\hat{U}) &= \int_{\underline{\theta}}^{\bar{\theta}} (v-\theta)[x(\theta) - \hat{x}(\theta)] + \Pi(U(\theta)) - \Pi(\hat{U}(\theta))\, dF(\theta) && (110) \\ &= \int_{\underline{\theta}}^{\bar{\theta}} \Pi(U(\theta)) - \Pi(\hat{U}(\theta))\, dF(\theta). && (111) \end{aligned}$$

Similarly to the argument at the end of the first step, (108) and (109) imply that $F^{\hat{U}}$ is a mean preserving spread of F^U, and hence (111) is positive, and this completes the proof. qed

Proof of Proposition 2 Note first if there is a solution $(x, U) \in \Lambda$ to the relaxed problem R_1, then because $(x, U) \in \Lambda$ is obviously feasible for the problem P_1', it is also a solution to P_1'. Moreover, any contract $(x, U) \in \Lambda$ satisfies the constraint IC^0 and thus a solution $(x, U) \in \Lambda$ to P_1' is also a solution to P. To see this, observe that for $(x, U) \in \Lambda$ we have that $\Theta_1^0 = (\theta_0, \bar{\theta}]$ by (36). To show IC^0, we thus have to show that $U(\theta) \geq U(\hat{\theta})$ for all $\hat{\theta} \in (\theta_0, \bar{\theta}]$ and $\theta \in \Theta$. But this is immediate from the definition of U in (36).

It remains to show existence of a solution $(x, U) \in \Lambda$ to R_1. For this recall that a cutoff contract is characterized by cutoffs $\theta_0 \in [\underline{\theta}, \bar{\theta}]$ and $U_0 \geq \theta_0 - \underline{\theta}$. We first show the auxiliary claim that for any $(\tilde{x}, \tilde{U}) \in \Lambda$ there is a $(x, U) \in \Lambda$ which yields a (weakly) higher profit than $(\tilde{x}, \tilde{U})$ and has the property that

$$U_0 \leq U^{SB} + (\bar{\theta} - \underline{\theta}). \tag{112}$$

Indeed, consider a $(\tilde{x}, \tilde{U})$ with cutoffs $(\tilde{\theta}_0, \tilde{U}_0)$ that violates (112). Since $(\tilde{x}, \tilde{U})$ is a cutoff contract,

[26]Given $\theta_0 = \hat{\theta}$, the cutoff U_0 exists by the intermediate value theorem, because the integral on the left hand side of (108) is strictly larger than the right hand side for $U_0 = \hat{U}(\underline{\theta})$, strictly lower for $U_0 = \hat{U}(v) + \hat{\theta} - \underline{\theta}$, and changes continuously in U_0.

this implies that $\tilde{U}(\theta) > U^{SB}$ for all θ. Define $(x, U) \in \Lambda$ with cutoffs

$$\theta_0 = \tilde{\theta}_0, \quad U_0 = \tilde{U}_0 - (\tilde{U}(\bar{\theta}) - U^{SB}). \tag{113}$$

By construction, we have that $U^{SB} \leq U(\theta) \leq \tilde{U}(\theta)$ for all θ. Thus, because Π is concave and uniquely maximized at U^{SB} by Lemma 3, this implies that $\Pi(U(\theta)) \geq \Pi(\tilde{U}(\theta))$ for all θ. Therefore, and since $x = \tilde{x}$, we obtain the profit

$$W(x, U) = \int (v - \theta)\tilde{x}(\theta) + \Pi(U(\theta))\, dF(\theta) \geq \int (v - \theta)\tilde{x}(\theta) + \Pi(\tilde{U}(\theta))\, dF(\theta) = W(\tilde{x}, \tilde{U}) \tag{114}$$

and this proves the auxiliary claim.

Now, let $\bar{\Lambda}$ be the set of cutoff contracts that satisfy (112). That is, $(x, U) \in \bar{\Lambda}$ if we can express (x, U) as a cutoff contract with cutoff $\theta_0 \in [\underline{\theta}, \bar{\theta}]$ and intercept $U_0 \in [\theta_0 - \underline{\theta}, U^{SB} + (\bar{\theta} - \underline{\theta})]$.

The auxiliary claim and Proposition 1 then imply that there is a solution $(x, U) \in \Lambda$ to R_1 if there is a solution to the problem

$$Q: \qquad \max_{(x,U)} W(x, U) \quad s.t. \quad (x, U) \in \bar{\Lambda}. \tag{115}$$

Because the profit $W(x, U)$ of a cutoff contract is pinned down by (θ_0, U_0), problem Q boils down to the problem of choosing a two-dimensional variable (θ_0, U_0) from the compact set $[\underline{\theta}, \bar{\theta}] \times [\theta_0 - \underline{\theta}, U^{SB} + (\bar{\theta} - \underline{\theta})]$. Because profit is continuous in (θ_0, U_0), there is a solution to Q. Therefore, there is a solution $(x, U) \in \Lambda$ to R_1, and this completes the proof. qed

References

Ashlagi, Itai, Constantinos Daskalakis, and Nima Haghpanah. 2023. Sequential Mechanisms With ex post Individual Rationality. Operations Research 71(1), 245-258.

Baron, David P., and David Besanko. 1984. Regulation and Information in a Continuing Relationship. Information Economics and Policy 1(3), 267-302.

Battaglini, Marco. 2005. Long-term Contracting with Markovian Consumers. American Economic Review 95(3), 637-658.

Bergemann, Dirk, Francisco Castro, and Gabriel Y. Weintraub. 2020. The Scope of Sequential Screening with Ex Post Participation Constraints. Journal of Economic Theory 188, 105055.

Celik, Gorkem. 2006. Mechanism Design With Weaker Incentive Compatibility Constraints. Games and Economic Behavior 56(1), 37-44.

Compte, Olivier, and Philippe Jehiel. 2009. Veto Constraint in Mechanism Design: Inefficiency with Correlated Types. American Economic Journal: Microeconomics 1(1), 182-206.

Daskalakis, Constantinos, Alan Deckelbaum, and Christos Tzamos. 2017. Strong Duality for a Multiple-Good Monopolist. Econometrica 85(3), 735-767.

Esö, Peter, and Balazs Szentes. 2017. Dynamic Contracting: An Irrelevance Theorem. Theoretical Economics 12(1), 109-139.

Harris, Milton, and Artur Raviv. 1979. Optimal Incentive Contracts with Imperfect Information." Journal of Economic Theory 20(2), 231-259.

Krähmer, Daniel, and Roland Strausz. 2015. Optimal Sales Contracts with Withdrawal Rights. The Review of Economic Studies 82(2), 762-790.

Krähmer, Daniel, and Roland Strausz. 2024. Unidirectional incentive compatibility. Mimeo.

Krasikov, Ilia, and Rohit Lamba. 2021. A Theory of Dynamic Contracting with Financial Constraints. Journal of Economic Theory 193, 105196.

Krasikov, Ilia, Lamba, Rohit, and Thomas Mettal. 2021. Implications of Unequal Discounting in Dynamic Contracting. Discussion Paper Penn State University.

Krishna, R. Vijay, Giuseppe Lopomo, and Curtis R. Taylor. 2013. Stairway to Heaven or Highway to Hell: Liquidity, Sweat Equity, and the Uncertain Path to Ownership. The RAND Journal of Economics 44(1), 104-127.

Manelli, Alejandro M., and Daniel R. Vincent. 2006. Bundling as an Optimal Selling Mechanism for a Multiple-Good Monopolist. Journal of Economic Theory 127(1), 1-35.

Mirrokni, Vahab, Renato Paes Leme, Pingzhong Tang, and Song Zuo. 2020. Non-Clairvoyant Dynamic Mechanism Design. Econometrica 88(5), 1939-1963.

Moore, John. 1984. Global Incentive Compatibility Constraints in Auction Design. Econometrica 52(6), 1523–1535.

Myerson, Roger B. 1986. Multistage Games with Communication. Econometrica 54(2), 323-358.

Pavan, Alessandro, Ilya Segal, and Juuso Toikka. 2014. Dynamic Mechanism Design: A Myersonian Approach. Econometrica 82(2), 601-653.

Riley, John, and Richard Zeckhauser. 1983. Optimal Selling Strategies: When to Haggle, when to Hold Firm. The Quarterly Journal of Economics 98(2), 267-289.

Rochet, Jean-Charles, and Philippe Choné. 1998. Ironing, Sweeping, and Multidimensional Screening. Econometrica 66(4), 783-826.

Samuelson, William. 1984. Bargaining under Asymmetric Information. Econometrica 52(4), 995-1005.

Sappington, David E.M. 1983. Limited liability Contracts Between Principal and Agent. Journal of Economic Theory 29(1), 1-21.

Spear, Stephen E., and Sanjay Srivastava. 1987. On Repeated Moral Hazard with Discounting. The Review of Economic Studies 54(4), 599-617.

Stokey, Nancy L., Robert E. Lucas, and Edward C. Prescott, 1989, Recursive Methods in Economic Dynamics. Cambridge Massachusetts, Harvard University Press.

Thomas, Jonathan, and Tim Worrall. 1990. Income Fluctuation and Asymmetric Information: An Example of a Repeated Principal-Agent Problem. Journal of Economic Theory 51(2), 367-390.