Logarithmic utility is a special case of CRRA utility

$f(a) = g(a) = 0$
L'Hôpital's rule applies:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Apply this tool to CRRA utility $\frac{c^{1-\sigma}-1}{1-\sigma}$. Let $\sigma \to 1$.

$$\lim_{\sigma \to 1} \frac{c^{1-\sigma} - 1}{1 - \sigma} = \lim_{\sigma \to 1} \frac{e^{\log\left(c^{1-\sigma}\right)} - 1}{1 - \sigma} = \lim_{\sigma \to 1} \frac{e^{(1-\sigma)\log(c)} - 1}{1 - \sigma}$$

Apply l'Hôpital's rule, taking derivatives of the numerator and of the denominator:

$$\lim_{\sigma \to 1} \frac{e^{(1-\sigma)\log(c)} - 1}{1 - \sigma} = \lim_{\sigma \to 1} \frac{e^{(1-\sigma)\log(c)} \cdot \left[-\log(c)\right]}{-1} = \log(c)$$